The Use of Gauss-Jordan Elimination Method in Determining the Proportion of Aggregate Gradation

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Abstract The determination of the proportion of combined aggregate gradation is an important step in the manufacture of asphalt mixtures, so this step needs to be performed carefully. Today, the determination of the proportion of combined aggregates is mainly done by trial and error or the graphical method, which often gives inaccurate results. In this study, the determination of the combined aggregate gradation proportion was carried out using the Gauss-Jordan elimination method to find the best solution by solving a set of linear equations formed from several aggregate grading groups. This study indicated that the Gauss-Jordan method provides good guidance in determining a single unique aggregate gradation proportion. The results of the iteration of this method can also show the trend of the characteristics of the combined gradation that can improve the understanding of the aggregate gradation used.

Keywords Gauss-Jordan elimination · Aggregate gradation · Combined aggregate proportion

1 Introduction

In the manufacture of hot mix asphalt, the determination of the proportion of the combined aggregate gradation is an important process, which can determine the properties of the resulting asphalt mixture. Determination of this proportion is generally carried out analytically or graphically. Many studies have been carried out related to this method of determining proportions [1–6]. The methods used include: (i) the simplest methods, namely the trial-and-error method; (ii) mathematical methods by substitution and elimination; (iii) the use of optimization techniques

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such as the least square method, linear and nonlinear programming; and (iv) graphical methods such as triangular or rectangular chart methods.

Each method has advantages and disadvantages, such as the trial-and-error method, which is the easiest but contains high uncertainty; the mathematical method is generally very accurate but often requires a reasonably long calculation time. In contrast, the graphical method has the advantage of being easy to work with, but the results are sometimes not accurate.

In this research, a mathematical method known as the Gauss-Jordan method is used to solve the blended aggregate problem, which is generally formulated in a set of linear equations with n unknowns. This method is one of the most accurate methods in determining the solution and is supported by the ease of operation so that the Gauss-Jordan method is expected to be widely used in determining the proportion of combined aggregates. In addition, this study also aims to evaluate the trend generated by a series of iterations of the Gauss-Jordan method on a specific aggregate gradation.

2 Gauss-Jordan Elimination Method

In determining the combined aggregate gradation, ones have two common options, i.e., using analytical methods or graphical methods. The analytical method, in the form of linear equations, generally is written Eqs. 1 and 2.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}.x_i = b_i \tag{1}$$

$$\sum_{i=j=1}^{n} a_{ij} = b_i \tag{2}$$

in which: a_{ij} is the ith percentage (by weight) of aggregate gradation j passing certain sieve size, x_i is the sought proportion of aggregate gradation j; b_i is the percentage (by weight) of combined aggregate retained on certain sieve size.

The value n represents the number of unknown, i.e., the proportion of aggregate gradation used to produce target gradation that complies with the specification. To determine a single unique solution of the unknown, one has to use the number of a linear equation similar to the number of unknown. Many techniques can solve a linear system problem; one of them is the Gauss elimination method. The Gauss elimination method systematically implements elementary row operations to a linear system to convert the system to upper triangular form and then back-substitute to obtain the solution. The improved method, namely Gauss-Jordan elimination, has increased the technique's effectiveness by conducting the elementary row operations to upper and lower triangular forms simultaneously.

Coupled with the use of a computer, it could be double the speed to obtain the solution.

Adenegan and Aluko [7] briefly provided that the procedure of Gauss Jordan elimination is as follows.

1. Write the normal matrix into the augmented matrix $[A/b]$.
2. Use elementary row operation on the augmented matrix $[A/b]$ to transform A into diagonal form.
3. Divide the diagonal elements and a right-hand side's element in each row by the diagonal elements in the row, and this will make the diagonal elements equal to one.

The problem encountered if there is a zero located on the diagonal of matrix A; then it could be switched the rows until a non-zero is in that element. If there is no possibility to find another non-zero, then stop because the system has either infinite or no solution.

The implementations of the above procedure to a 4×4 matrix are shown in Eqs. 3–7.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{3}$$

Step a The matrix in Eq. 3 can be written in an augmented matrix [A/b] as shown in Eq. 4.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right) \tag{4}$$

Step b Use the elementary row operation on the matrix [A/b] to make the diagonal of the matrix equals to one. To do so, eliminate x_1 from the 2nd, 3rd, and 4th row by subtracting the element on that row with the multiplication of the element x_1 from the 1st row and the following ratio $m_{21} = a_{21}/a_{11}$, $m_{31} = a_{31}/a_{11}$, and $m_{41} = a_{41}/a_{11}$, respectively (Eq. 5).

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & a_{34}^{(2)} & b_3^{(2)} \\ 0 & a_{42}^{(2)} & a_{43}^{(2)} & a_{44}^{(2)} & b_4^{(2)} \end{array}\right) \tag{5}$$

in which: $a_{22}^{(2)} = a_{22} - a_{12} \,.\, m_{21}$; $b_2^{(2)} = b_2 - b_1 \,.\, m_{21}$; the other elements follow similar calculation pattern.

The subsequent elimination is for x_2, especially on the first, third, and fourth rows by subtracting the element on that rows with the multiplication of the element x_2 from the 2nd row and the ratio $m_{12} = a_{12}/a_{22}$, $m_{32} = a_{32}/a_{22}$, and $m_{42} = a_{42}/a_{22}$, respectively (Eq. 6).

$$\left(\begin{array}{cccc|c} a_{11}^{(3)} & 0 & a_{13}^{(3)} & a_{14}^{(3)} & b_1^{(3)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & b_3^{(3)} \\ 0 & 0 & a_{43}^{(3)} & a_{44}^{(3)} & b_4^{(3)} \end{array}\right) \tag{6}$$

where $\mathrm{a}_{33}^{(3)} = \mathrm{a}_{33}^{(2)} - \mathrm{a}_{23}.\ \mathrm{m}_{32}$; $\mathrm{b}_1^{(3)} = \mathrm{b}_1 - \mathrm{b}_2^{(2)}.\ \mathrm{m}_{12}$; the others follow similar calculation pattern.

Repeat this elimination procedure until only the diagonal elements of A contains non-zero elements.

Step c Divide the non-zero elements and the corresponding b element on each row with the diagonal element on that row (Eq. 7).

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1/a_{11} \\ 0 & 1 & 0 & 0 & b_2/a_{22} \\ 0 & 0 & 1 & 0 & b_3/a_{33} \\ 0 & 0 & 0 & 1 & b_4/a_{44} \end{array}\right) \tag{7}$$

3 Research Methodology

The methodology of this work can be summarized into four steps as follows.

1. Preparing and testing the aggregate properties
   Four groups of aggregate, i.e., two coarse-aggregates and two fine-aggregates, were collected from a local quarry, and they were tested to determine the properties before they were sieved to know the gradation. The properties of the aggregate were tested based on the Indonesian National Standard (SNI) and the results obtained have to comply with the specification of Directorate General of Highways year 2018 version 2 [8].
2. Sieving the aggregate materials
   Once the properties of the aggregates can fulfill the specification, the aggregates were prepared for sieving. In this work, the aggregate was assumed to be used for asphalt mixture AC-WC (asphalt cours—wearing course). Each group of aggregates was a sieve, and the result was checked against the specification of Directorate General of Highways year 2018 version 2 [8].

3. Conducting analysis using Gauss-Jordan elimination method
   Four aggregate gradations were blended to obtain the target gradation; therefore, four unknowns were to be solved, i.e., x_1 to x_4. According to Eqs. 1 and 2, three equations were selected based on Eq. 1, and the last equation adopted Eq. 2. There were nine sieve sizes, and three sieve sizes were used in each Gauss-Jordan elimination iteration; therefore, by using the combination formula, there were 84 Gauss-Jordan elimination iterations.
4. Evaluating the results
   The analysis results were evaluated by considering whether the target gradation can be achieved for each aggregate proportion produced and the trend of the combined gradation against the specification used.

4 Results and Analysis

The aggregate is required to be tested first to ensure that the aggregate can be used for the next step. As mentioned in Sect. 3, there were four groups of aggregate, i.e., two of them were coarse aggregate (Grad-A and Grad-B), and the rest were fine aggregate (Grad-C and Grad-D). The results of the property test are shown in Table 1.

Table 1 shows that all aggregate properties could fulfill the specification used, the Specification of Directorate General of Highways Year 2018 ver. 2 [8]. It means that both coarse and fine aggregates can be used for the next test, i.e., sieve test.

The asphalt mixture used in this study, i.e., asphalt concrete—wearing course (AC-WC), is a mixture with a dense-graded aggregate gradation. It requires ten

Table 1 Results of the property tests of the aggregate

Types of aggregate	Property of aggregate	Results	Specification
Coarse aggregate	Los Angeles abrasion value	14.59	Min. 40%
	Bulk specific gravity of aggregate of Grad-A	2.667	Min. 2.5
	Bulk specific gravity of aggregate of Grad-B	2.634	Min. 2.5
	Absorption of aggregate of Grad-A	1.209	Max. 3%
	Absorption of aggregate of Grad-B	1.406	Max. 3%
Fine aggregate	Bulk specific gravity of aggregate of Grad-C	2.611	Min. 2.5
	Bulk specific gravity of aggregate of Grad-D	2.617	Min. 2.5
	Absorption of aggregate of Grad-C	1.922	Max. 3%
	Absorption of aggregate of Grad-D	1.096	Max. 3%

Table 2 Different aggregate gradations were used in this study

Sieve sizes		Percentage of passing sieve size				
No.	mm	Grad-A	Grad-B	Grad-C	Grad-D	Spec
¾″	19	100	100	100	100	100
½″	12.5	28.71	100	100	100	90–100
3/8″	9.5	3.35	90.36	100	100	77–90
4	4.75	0.76	23.94	100	100	53–69
8	2.36	0.56	3.70	82.80	90.23	33–53
16	1.18	0.48	2.10	55.34	74.31	21–40
30	0.6	0.44	1.79	34.68	55.10	14–30
50	0.3	0.41	1.62	24.08	36.23	9–22
100	0.15	0.33	1.23	11.94	20.28	6–15
200	0.075	0.24	0.84	6.72	6.00	4–9

aggregate sieve sizes with the maximum aggregate size is no. ¾″ (or the nominal maximum aggregate size/NMAS is ½″). The results of the sieve test for four groups of aggregates (Grad-A to Grad-D) are depicted in Table 2.

Table 2 shows that none of the groups of aggregate could fulfill the specification. It is indicated that a blend of that four gradations was required to obtain the target gradation that complies with the specification. It is possible to determine the proportion of the four gradations by using the trial-and-error method, but for a particular case, it is not easy to find the exact solution and is also time-consuming.

In this work, the Gauss-Jordan elimination method is proposed. The work only needs simple software such as an Excel spreadsheet. It has to be admitted that this method required much work in the beginning. It included: (i) developing the template of the calculation of the Gauss-Jordan method using Eqs. 4–7. It needed three linear equations of Eq. 1 selected in a systematic way among nine possible percentages of aggregate passing certain sieve size (Table 2) and one another equation of Eq. 2; (ii) calculating the combined gradation using the proportion xi produced by point (i); and (iii) checking whether the target gradation is achieved by examining the percentage of aggregate passing of each sieve size against the specification. It is beneficial if all calculation cells in the spreadsheet are linked to the master table (see Table 2) because the proportion calculation will be conducted directly if there is an update on the master table in the future.

As mentioned in Sect. 3, Gauss-Jordan elimination was iterated as many as 84 times in this work to enable checking the possibility of the combination of three among nine aggregate sieve sizes to produce a correct proportion. The selection of three aggregate sizes to form a combination was conducted systematically to make it easier for the analysis. A result indicated a criterion of a correct proportion that none of the proportions was less than zero and more than 1. Among all possible combinations of aggregate sieve sizes in this work, only 15 combined gradations can produce correct proportions. Together with some incorrect proportions (i.e., negative proportions and proportion more than 1, indicated by bold letters), some of them were selected randomly and presented in Tables 3 and 4. In Tables 3 and 4, all proportion is denoted in percentage unit.

The combined gradation (stated as CGrad) in Tables 3 and 4 are divided into two parts: the combined gradation with correct proportions (CGrad-4 to CGrad-7) and the rest are the combined gradation with incorrect proportions (CGrad-1 to CGrad-3, and CGrad-8 to CGrad-10). The CGrads on the left of the table are the ones with the most combinations of coarse aggregate; the CGrads on the right of the table are the ones with the most combinations of fine aggregates. The combined gradation with correct proportions generally has a relative balance of a combination of coarse and fine aggregates.

Beside correct proportions, the target gradations have to produce a good result, i.e., it could fulfill the specification. Unfortunately, as seen in the tables, none of the combined gradation can produce good results. The four gradations with correct proportions (CGrad-4 to CGrad-7) in the table are the ones that has less deviation against the specification.

It is not too surprising to know this result because all combined gradations obtained in this work are only a small part of a substantial possible combination that could be produced by using the analytical method. It means that Gauss-Jordan

Table 3 The five combined gradations were used firstly in this study

Sieve No.	Percentage passing sieve size					
	CGrad-1	CGrad-2	CGrad-3	CGrad-4	CGrad-5	Specification
	$x_1 = 7$	$x_1 = 7$	$x_1 = 7$	$x_1 = 7$	$x_1 = 7$	
	$\boldsymbol{x_2 = 101}$	$\boldsymbol{x_2 = 101}$	$x_2 = 42$	$x_2 = 42$	$x_2 = 44$	
	$\boldsymbol{x_3 = -623}$	$\boldsymbol{x_3 = -120}$	$x_3 = 60$	$x_3 = 33$	$x_3 = 35$	
	$x_4 = 616$	$\boldsymbol{x_4 = 112}$	$\boldsymbol{x_4 = -10}$	$x_4 = 18$	$x_4 = 14$	
¾″	100.0	100.0	100.0	100.0	100.0	100
½″	95.0	95.0	95.0	95.0	95.0	90–100
3/8″	83.5	83.5	89.2	89.2	88.9	77–90
4	16.3	16.3	61.0	61.0	59.3	53–69
8	43.0	5.6	43.0	45.0	43.0	33–53
16	114.6	19.1	27.2	32.4	30.5	21–40
30	124.8	22.0	16.5	22.0	20.5	14–30
50	74.6	13.4	11.8	15.1	14.1	9–22
100	51.7	9.7	5.8	8.1	7.5	6–15
200	−4.1	−0.5	3.9	3.7	3.6	4–9

CGrad-1, CGrad-2, CGrad-3, CGrad-4, and CGrad-5 were determined using all gradation with sieve no. ½″, 3/8″, no. 8; ½″, 3/8″, no. 30; ½″, no. 4, no. 8; ½″, no. 4, no. 30; and ½″, no. 8, no. 16, respectively

Table 4 The other combined gradations were used in this study

Sieve No.	Percentage passing sieve size					
	CGrad-6	CGrad-7	CGrad-8	CGrad-9	CGrad-10	Specification
	x_1 = 14	x_1 = 13	$\boldsymbol{x_1}$ **= −4**	$\boldsymbol{x_1}$ **= −37**	$\boldsymbol{x_1}$ **= −196**	
	x_2 = 33	x_2 = 45	x_2 = 57	x_2 = 99	$\boldsymbol{x_2}$ **= 256**	
	x_3 = 51	x_3 = 11	x_3 = 25	$\boldsymbol{x_3}$ **= −4**	x_3 = 19	
	x_4 = 2	x_4 = 31	x_4 = 22	x_4 = 41	x_4 = 21	
¾″	100.0	100.0	100.0	100.0	100.0	100
½″	90.2	91.0	103.1	126.1	239.9	90–100
3/8″	83.5	83.5	98.7	125.8	264.9	77–90
4	61.0	53.4	61.0	61.0	99.8	53–69
8	45.3	39.3	43.0	37.6	43.0	33–53
16	30.5	30.5	31.6	30.5	30.5	21–40
30	19.5	22.0	22.0	23.0	21.8	14–30
50	13.6	14.8	15.0	15.5	15.5	9–22
100	7.0	8.3	8.2	9.0	9.0	6–15
200	3.9	3.0	3.5	3.0	4.2	4–9

CGrad-6, CGrad-7, CGrad-8, CGrad-9, and CGrad-10 were determined using all gradation with sieve no. and 3/8″, no. 4, no. 16; 3/8″, no. 16, no. 30; no. 4, no. 8, no. 30; no. 4, no. 16, no. 50; and no. 4, no. 16, no. 50, respectively

elimination is only a method that assists ones in indicating correct proportions. If the proportion(s) cannot produce the target gradation that fulfills the specification, an adjustment has to be conducted by changing the proportion slightly based on the combined gradation obtained. For example, CGrad-6 is the best combined gradation produced by Gauss-Jordan in this study, although the gradation cannot produce a good result. By changing x_1 from 14 to 12%, and x_3 from 51 to 53% while the rest

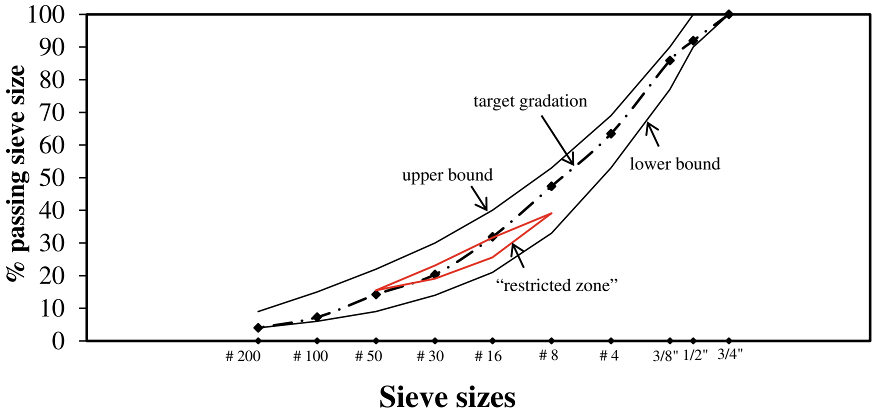

Fig. 1 Plotting of the selected combined gradation against the specification

remains the same, all combined gradation now can fulfill the specification. Further examination can be carried out by plotting the combined gradation produced using correct proportions against the specification (see Fig. 1).

Although "restricted zone" (the area with the red line in Fig. 1) is omitted in Indonesian specification, the knowledge of this area is useful to help one in determining a better aggregate gradation. According to Al-Khateeb et al. [9], the crossover-through-restricted-zone curve (the target gradation in Fig. 1) could be more susceptible to permanent deformation (rutting). A below-restricted-zone curve is preferable to provide sufficient voids for a more long-lasting asphalt mixture.

In addition, using many iterations of Gauss-Jordan, one also can analyze the trend produced. For example, it is more challenging to obtain target gradation with the percentage of coarse aggregate higher than that of fine aggregate. The combined gradation with higher coarse aggregate proportions or $x_1 + x_2 > x_3 + x_4$ (i.e., CGrad-5 and CGrad-7) has a lower percentage passing sieve no. 200 than those of gradation with higher fine aggregates (i.e., CGrad-4 and CGrad-6).

5 Conclusions

This paper presented a systematical method, i.e., Gauss-Jordan elimination, to determine the proportion of aggregate gradation mathematically. To do so, a dense graded aggregate gradation consisted of four groups of aggregate was used in this study. Eighty-four iteration was conducted by systematically selecting three of nine possible aggregate sieve sizes plus one proportion summation to produce four linear equation systems that will be solved using Gauss-Jordan elimination. The results indicated that the Gauss-Jordan method could produce many benefits in solving the blended aggregate problems. Although this method did not always produce a good result that complies with the specification, the solution obtained can still be used as guidance to determine a single unique solution of the aggregate proportion. In addition, the iteration of the method may also show the trend of the characteristics of the combined gradation that can improve the understanding of the aggregate gradation used.

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