Properties of Bounded Variation Function of Two Variables

by Susilo Hariyanto

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Abstract. Function of bounded variation of one variable was first introduced by Camille Jordan in 1881. Functions of bounded variation is a function with finite variation. The method used in this research is literature study, namely by studying and understanding about the concept of function of bounded variation in $\mathbb{R}$, and then expand the concept into the space $\mathbb{R}^2$. This study aims to complete the evidence of the properties of a function of bounded variation of two variables. The results obtained from this study are that each the function of bounded variation of two variables is a finite function. This research also discusses the relationship between the function of bounded variation and monotonous functions. The function of bounded variation of two variables is also the finite function in each sub square subset of $\mathbb{R}^2$. Each variation of the function of bounded variation can be expressed as the sum of its sub-squares.

INTRODUCTION

One of the basic concepts in mathematical analysis is function. Given two non-empty sets A and B, a function from set A to set B is the mapping of each member of set A with exactly one member of set B. The function of set A to set B is denoted by $f: A \to B$.

The study of functions is increasingly developing along with the large number of studies conducted by mathematicians. One of the topics of functions being developed is bounded variation functions. Bounded variation functions of one variable were first introduced by Camille Jordan in 1881.

After Camille Jordan studied bounded variation function of one variable, many researchers who studied bounded variation functions of two variables, among them Clarkson and Adams [1] have discussed seven definitions of generalization bounded variation function for two variables. From these definitions, there are two definitions used in this study. The two definitions are known as the Vitali variation and the Hardy-Krause variation, this concept previously discussed by Owen [4]. Adams and Clarkson also discussed the properties of a bounded variation functions for two variables [2]. In addition, Azocar et al. Also discussed the space of bounded variation functions of two-variables with square domain in space $\mathbb{R}^2$ [3]. In 1994, Dariusz Idczak [5] also studied the bounded variation function for several variables and their differentiability. This research is discussed in space $\mathbb{R}^2$ with interval function domain $[0,1] \times [0,1] = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1 \}$.

In this study, the authors aim to examine definition of the variation of the function in a non-empty square subset of $\mathbb{R}^2$ and complete the evidence for the properties of a bounded variation function for two variables.

PRELIMINARIES, BACKGROUND, NOTATIONS

We start with some of the definitions used in this study. The first one describes the bounded function, bounded variation function of one variable and the partition of an interval.

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Definition 2.1 [6] *Let A is a set. A function $g: A \to \mathbb{R}$ is said to be bounded to A if there is a constant $K > 0$ such that $|g(x)| \leq K$ for all $x \in A$.*

The partition of an interval is defined as follows.

Definition 2.2 [8] *The partition of an interval $[a, b]$ is a non-overlapping collection of $P = \{I_1, I_2, \dots, I_n\}$ whose the union is the interval $[a, b]$. The n-th partition of the interval $[a, b]$ can be written as $I_n = [x_{n-1}, x_n]$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.*

From the above definition, it can be defined the variation of functions and bounded variation function of one variable.

Definition 2.3 [8] *Given a function $f: [a, b] \to \mathbb{R}$ and $[u, v] \subseteq [a, b]$.. The variation of the function f on $[u, v]$ is denoted by $V(f, [u, v])$ and is defined as follows*

$$V(f, [u, v]) = \sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| | x_i: 1 \leq i \leq n \text{ partition of } [u, v]\right\} \tag{1}$$

Where the supremum is taken from all possible partitions on $[u, v]$. The function f is said to be bounded variation function if $V(f, [u, v]) < \infty$.

Next, we examined the relation of bounded variation functionand the monotone function.

Theorem 2.4 [7]

- *If f is increasing on $[a, b]$, then f is of bounded variation on $[a, b]$ and $V(f, [a, b]) = f(b) - f(a)$.*
- *If f is increasing on $[a, b]$, then f is of bounded variation on $[a, b]$ and $V(f, [a, b]) = f(a) - f(b)$.*

Furthermore, it is explained the theorem which states that each bounded variation function of one variable is also bounded to each subinterval and the value of the variation can be expressed as the sum of the variations of each subinterval.

Theorem 2.5 [7] *Given a function $f: [a, b] \to \mathbb{R}$, and let $c \in (a, b)$. If a function f is bounded variation in $[a, c]$ and $[c, b]$, then f is bounded variation in $[a, b]$ and*

$$V(f, [a, b]) = V(f, [a, c]) + V(f, [c, b]). \tag{2}$$

Research of bounded variation function begins with understanding the definition of a partition in space $\mathbb{R}^2$. Taken any square $I_{\boldsymbol{a}}^{\boldsymbol{b}} = I \times J$ subset of space $\mathbb{R}^2$ where $I = [x_1, x_2]$ and $J = [y_1, y_2]$ interval in $\mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$, and $\boldsymbol{a} = (x_1, y_1)$, $\boldsymbol{b} = (x_2, y_2)$ vector in $\mathbb{R}^2$.

Definition 2.6 [1] *Partition of square $I_{\boldsymbol{a}}^{\boldsymbol{b}} \subset \mathbb{R}$ is defined by*

$$Q = \{[t_i, t_{i+1}] \times [s_j, s_{j+1}] | i = 1,2, \dots, m \text{ dan } j = 1,2, \dots, n\}, \tag{3}$$

where $\{t_i | i = 1,2, \dots m\}$ is a partition of interval I, and $\{s_j | j = 1,2, \dots, n\}$ is a partition of interval J.

Suppose that $P(I)$ represents a collection of all partition of interval I, then $\{t_i | i = 1,2, \dots m\} \in P(I)$, and $P(J)$ represents of all partitions of interval J, then $\{s_j | j = 1,2, \dots, n\} \in P(J)$.

If we define function $f: I_a^b \to \mathbb{R}$, then it applies

$$\begin{aligned} \Delta_{10} f(t_{i+1}, s_{j+1}) &= f(t_{i+1}, s_{j+1}) - f(t_i, s_{j+1}), \\ \Delta_{01} f(t_{i+1}, s_{j+1}) &= f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j), \\ \Delta_{11} f(t_{i+1}, s_{j+1}) &= \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j)\right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j)\right). \end{aligned} \tag{4}$$

From the definition of partitions in space $\mathbb{R}^2$ above, it can be further defined that variation functions and bounded variation functions in space $\mathbb{R}^2$.

Definition 2.7 [3] *Given a function $f: I_{\boldsymbol{a}}^{\boldsymbol{b}} \to \mathbb{R}$, and $y \in J$ be fixed, then the Jordan variation of the function $f(.,y): I \to \mathbb{R}$ is defined by*

$$V_I(f(.,y)) = \sup_{P(I)} \sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1}, y)|, \tag{5}$$

Where supremum is taken from all partition of $P(I) = \{t_i | i = 1,2, \dots, m\}$ in interval I, and

For any $x \in I$ be fixed, then the Jordan variation of the function $f(x,.): J \to \mathbb{R}$ is defined by

$$V_J(f(x,.)) = \sup_{P(J)} \sum_{j=1}^{n-1} |\Delta_{01} f(x, s_{j+1})|, \tag{6}$$

Where supremum is taken from all partition of $P(J) = \{s_j | j = 1,2, \dots, n\}$ in interval J, and

So, the total variation of the function f is defined by

$$TV(f, I_a^b) = V_I(f(.,y)) + V_J(f(x,.)) + V_{I_a^b}(f). \tag{7}$$

The function f is said to be bounded variation function if the total variastion of the function f is finite, or

$$TV(f, I_a^b) < \infty. \tag{8}$$

The set of the bounded variation function of I_a^b is denoted by $BV(I_a^b)$.

MAIN RESULT

In this research we discuss about the properties of the bounded variation function of two variables and we give the proofs. First, we examined the relation of bounded variation function with bounded function.

Theorem 3.1 Let the function $f: I_a^b \to \mathbb{R}$ be bounded variation in I_a^b, then f be bounded in I_a^b.

Proof. we know that f be bounded variation in I_a^b, then

$$\begin{aligned} TV(f, I_a^b) &= V_I(f(.,y)) + V_J(f(x,.)) + V_{I_a^b}(f) \\ &= M + K + N < \infty. \end{aligned} \tag{9}$$

we will proof that f be bounded in I_a^b.

1. for $V_I(f(.,y))$, there exist $M > 0$ such that $|f(t,y)| < M$, for every $t \in I = [x_1, x_2]$, we get

$$|f(t,y) - f(x_1,y)| + |f(x_2,y) - f(t,y)| \leq V_I(f(.,y)), \tag{10}$$

where $|f(x_1,y)|$ and $|f(x_2,y)|$ be bounded on $I = [x_1, x_2]$. Such that

$$\begin{aligned} |f(t,y)| &= |f(t,y) - f(x_1,y) + f(x_1,y)| \\ &\leq |f(t,y) - f(x_1,y)| + |f(x_1,y)| \\ &\leq |f(t,y) - f(x_1,y)| + |f(t,y) - f(x_2,y)| + |f(x_1,y)| + |f(x_2,y)| \\ &\leq V_I(f(.,y)) + |f(x_1,y)| + |f(x_2,y)| \\ &= M < \infty. \end{aligned} \tag{11}$$

2. for $V_J(f(x,.))$, there exist $K > 0$ such that $|f(x,s)| < K$, for every $s \in J = [y_1, y_2]$, then the next step is similarity with point 1.

3. for $V_{I_a^b}(f)$, there exist $N > 0$ such that $|f(t,s)| < N$, for every $(t,s) \in I_a^b = [x_1,x_2] \times [y_1,y_2]$, we get

$$\left|f(t,s) - \left(f(x_1,y_2) - f(x_1,y_1)\right)\right| + |(f(x_2,y_2) - f(x_2,y_1)) - f(t,s)| \leq V_{I_a^b}(f), \tag{12}$$

where $|f(x_1,y_1)|$, $|f(x_1,y_2)|$, $|f(x_2,y_1)|$ and $|f(x_2,y_2)|$ be bounded on $I_a^b = [x_1,x_2] \times [y_1,y_2]$. Such that

$$\begin{aligned} |f(t,s)| &= \left|f(t,s) - \left(f(x_1,y_2) - f(x_1,y_1)\right) + \left(f(x_1,y_2) - f(x_1,y_1)\right)\right| \\ &\leq \left|f(t,s) - \left(f(x_1,y_2) - f(x_1,y_1)\right)\right| + \left|\left(f(x_1,y_2) - f(x_1,y_1)\right)\right| \\ &\leq \left|f(t,s) - \left(f(x_1,y_2) - f(x_1,y_1)\right)\right| + \left|f(t,s) - \left(f(x_2,y_2) - f(x_2,y_1)\right)\right| + \left|\left(f(x_1,y_2) - f(x_1,y_1)\right)\right| \\ &\quad + |(f(x_2,y_2) - f(x_2,y_1))| \\ &\leq V_{I_a^b}(f) + \left|\left(f(x_1,y_2) - f(x_1,y_1)\right)\right| + |(f(x_2,y_2) - f(x_2,y_1))| \\ &= N < \infty. \end{aligned} \tag{13}$$

from point 1, 2, and 3 it can be concluded that

$$|f(x,y)| = M + K + N < \infty. \tag{14}$$

for every $x, y \in I_a^b = [x_1,x_2] \times [y_1,y_2]$. So we get that f be bounded.

for the next we will discuss the relation between the bounded variation function and monotonous functions. We defined the functions $f: I_a^b = [x_1,x_2] \times [y_1,y_2] \to \mathbb{R}$ is the strictly increasing functions if $f(.,y): [x_1,x_2] \to \mathbb{R}$ is increasing for any $y \in [y_1,y_2]$ and $f(x,.): [y_1,y_2] \to \mathbb{R}$ is increasing for any $x \in [x_1,x_2]$.

Theorem 3.2 *If the function f is strictly increasing in I_a^b, then the function f is bounded variation in I_a^b and*

4. $V_I(f(.,y)) = f(x_2,y) - f(x_1,y)$,
5. $V_J(f(x,.)) = f(x,y_2) - f(x,x_1)$,
6. $V_{I_a^b}(f) = \left(f(x_2,y_2) - f(x_2,y_1)\right) - \left(f(x_1,y_2) - f(x_1,y_1)\right)$. (15)

Proof.

1. We have the function f is strictly increasing in I_a^b, its meaning that for every $y \in J = [y_1,y_2]$, $f(.,y)$ is strictly increasing in interval $I = [x_1,x_2]$ then for every $y \in J = [y_1,y_2]$, so that

$$V_{[x_1,x_2]}(f(.,y)) = f(x_2,y) - f(x_1,y), \tag{16}$$

Let $\{t_i | i = 1, \ldots, m\}$ be a partition in interval I with $t_i < t_{i+1}$ and $t_1 = x_1$ also $t_m = x_2$, we get

$$\sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1}, y)| = \sum_{i=1}^{m-1} |f(t_{i+1}, y) - f(t_i, y)|$$

$$= |f(t_2,y) - f(t_1,y)| + |f(t_3,y) - f(t_2,y)| + \cdots + |f(t_m,y) - f(t_{m-1},y)|. \quad (17)$$

Therefore $t_i < t_{i+1}$ and f is increasing so $f(t_i,y) < f(t_{i+1},y)$ for $i = 1,2,\dots,m$. Consider,

$$f(t_{i+1},y) - f(t_i,y) > 0$$
$$|f(t_{i+1},y) - f(t_i,y)| = f(t_{i+1},y) - f(t_i,y). \quad (18)$$

Because the function f is strictly increasing in I_a^b dan $x_1 = t_1 < t_2 < \cdots < t_m = x_2$, then

$$V_I(f(.,y)) = \sup_{P(I)} \sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1},y)|$$
$$= \sup_{P(I)} \sum_{i=1}^{m-1} |f(t_{i+1},y) - f(t_i,y)|$$
$$= |f(t_2,y) - f(t_1,y)| + |f(t_3,y) - f(t_2,y)| + \cdots + |f(t_m,y) - f(t_{m-1},y)|$$
$$= f(t_2,y) - f(t_1,y) + f(t_3,y) - f(t_2,y) + \cdots + f(t_m,y) - f(t_{m-1},y)$$
$$= f(t_m,y) - f(t_1,y), \quad (19)$$

Notice that $t_1 = x_1$ and $t_m = x_2$, such that

$$V_I(f(.,y)) = f(x_2,y) - f(x_1,y). \quad (20)$$

So, we get $V_I(f(.,y)) = f(x_2,y) - f(x_1,y) < \infty$, then $V_I(f(.,y))$ bounded in I_a^b.

2. Similarly, if f strictly increasing in interval J then

$$V_J(f(x,.)) = f(x,y_2) - f(x,y_1). \quad (21)$$

3. We have the function f is strictly increasing in I_a^b, its meaning that for every $x_1,x_2,y_1,y_2 \in I_a^b = [x_1,x_2] \times [y_1,y_2]$, the function f is increasing in I_a^b then for any $x_1,x_2,y_1,y_2 \in I_a^b = [x_1,x_2] \times [y_1,y_2]$, we have

$$V_{I_a^b}(f) = \big(f(x_2,y_2) - f(x_2,y_1)\big) - \big(f(x_1,y_2) - f(x_1,y_1)\big). \quad (22)$$

Let $Q = \{[t_i,t_{i+1}] \times [s_j,s_{j+1}] \,|\, i = 1,\dots,m \text{ dan } j = 1,\dots,n\}$ be a partition in I_a^b with $(t_i,s_j) < (t_{i+1},s_{j+1})$ and $t_1 = x_1$, $t_m = x_2$, $s_1 = y_1$, also $s_n = y_2$, we get

$$\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} |\Delta_{11} f(t_{i+1},s_{j+1})| = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \Big|\big(f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j)\big) - \big(f(t_i,s_{j+1}) - f(t_i,s_j)\big)\Big|$$
$$= \big|\big(f(t_2,s_2) - f(t_2,s_1)\big) - \big(f(t_1,s_2) + f(t_1,s_1)\big)\big|$$
$$+ \big|\big(f(t_3,s_3) - f(t_3,s_2)\big) - \big(f(t_2,s_3) + f(t_2,s_2)\big)\big| + \cdots$$
$$+ \big|\big(f(t_m,s_n) - f(t_m,s_{n-1})\big) - \big(f(t_{m-1},s_n) + f(t_{m-1},s_{n-1})\big)\big|. \quad (23)$$

Because of $(t_i,s_j) < (t_{i+1},s_{j+1})$ and f is increasing then $f(t_i,s_j) < f(t_{i+1},s_{j+1})$ for $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$.

So,

$$\big(f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j)\big) - \big(f(t_i,s_{j+1}) - f(t_i,s_j)\big) > 0$$
$$\Big|\big(f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j)\big) - \big(f(t_i,s_{j+1}) - f(t_i,s_j)\big)\Big|$$
$$= \big(f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j)\big) - \big(f(t_i,s_{j+1}) - f(t_i,s_j)\big). \quad (24)$$

Consider that f is increasing in I_a^b and $x_1 = t_1 < t_2 < \cdots < t_m = x_2$ also $y_1 = s_1 < s_2 < \cdots < s_n = y_2$, we get

$$V_{I_a^b}(f) = \sup_Q \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} |\Delta_{11} f(t_{i+1},s_{j+1})|$$
$$= \sup_Q \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} \Big|\big(f(t_{i+1},s_{j+1}) - f(t_{i+1},s_j)\big) - \big(f(t_i,s_{j+1}) - f(t_i,s_j)\big)\Big|$$
$$= \big|\big(f(t_2,s_2) - f(t_2,y_1)\big) - \big(f(x_1,s_2) - f(x_1,y_1)\big)\big| + \big|\big(f(t_3,s_3) - f(t_3,s_2)\big) - \big(f(t_2,s_3) - f(t_2,s_2)\big)\big| + \cdots$$
$$+ \big|\big(f(x_2,y_2) - f(x_2,s_{n-1})\big) - \big(f(t_{m-1},y_2) - f(t_{m-1},s_{n-1})\big)\big|$$
$$= f(t_2,s_2) - f(t_2,y_1) - f(x_1,s_2) + f(x_1,y_1) + f(t_3,s_3) - f(t_3,s_2) - f(t_2,s_3) + f(t_2,s_2) + \cdots \quad (25)$$
$$+ \big(f(x_2,y_2) - f(x_2,s_{n-1})\big) - f(t_{m-1},y_2) + f(t_{m-1},s_{n-1}).$$

therefore $t_1 = x_1$, $t_m = x_2$, $s_1 = y_1$, and $s_n = y_2$, so

$$V_{I_a^b}(f) = \big(f(x_2,y_2) - f(x_2,y_1)\big) - \big(f(x_1,y_2) - f(x_1,y_1)\big). \quad (26)$$

We have $V_{I_a^b}(f) = \big(f(x_2,y_2) - f(x_2,y_1)\big) - \big(f(x_1,y_2) - f(x_1,y_1)\big) < \infty$, then $V_{I_a^b}(f)$ bounded in I_a^b.

Next, the theorem is presented which states that if a bounded variation function to an interval, then the variation is bounded to each subinterval and the variation of its function can be expressed as the sum of the variations of each subinterval.

Theorem 3.3 *Let the function* $f: I_a^b \to \mathbb{R}$, $y \in J$ *be fixed and* $f(.,y): I = [x_1,x_2] \to \mathbb{R}$, *let* $c \in [x_1,x_2]$. *If* $f(.,y)$ *bounded variation in* $[x_1,c]$ *and* $[c,x_2]$, *then* $f(.,y)$ *is bounded variation in* $[x_1,x_2]$ *and*

$$V_I(f(.,y)) = V_{[x_1,c]}(f) + V_{[c,x_2]}(f). \tag{27}$$

Proof. Taken any partition $P(I) = \{t_i | i = 1, \dots, m\}$ in interval $I = [x_1,x_2]$, let $c \in [x_1,x_2]$ and $t_{k-1} < c < t_k$, for $k \in \{1,2,\dots,m\}$, such that

$$\begin{aligned}\sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1},y)| &= \sum_{i=1}^{m-1} |f(t_{i+1},y) - f(t_i,y)| \\ &= \sum_{i=1}^{k-2} |f(t_{i+1},y) - f(t_i,y)| + |f(t_k,y) - f(t_{k-1},y)| + \sum_{i=k}^{m-1} |f(t_{i+1},y) - f(t_i,y)| \\ &= \sum_{i=1}^{k-2} |f(t_{i+1},y) - f(t_i,y)| + |f(t_k,y) - f(c,y) + f(c,y) - f(t_{k-1},y)| \\ &\quad + \sum_{i=k}^{m-1} |f(t_{i+1},y) - f(t_i,y)|.\end{aligned} \tag{28}$$

therefore $t_{k-1} < c < t_k$, so

$$\begin{aligned}&\le \sum_{i=1}^{k-2} |f(t_{i+1},y) - f(t_i,y)| + |f(c,y) - f(t_{k-1},y)|| + |f(t_k,y) - f(c,y)| \\ &\quad + \sum_{i=k}^{m-1} |f(t_{i+1},y) - f(t_i,y)|.\end{aligned} \tag{29}$$

Because the partition $\{t_1, t_2 \dots, t_{k-1}, c\}$ lies at interval $[x_1,c]$ and the partition $\{c, t_k, t_{k+1}, \dots, t_m\}$ lies at interval $[c,x_2]$, such that

$$\begin{aligned}&\le \sum_{i=1}^{k-2} |f(t_{i+1},y) - f(t_i,y)| + |f(c,y) - f(t_{k-1},y)|| + |f(t_k,y) - f(c,y)| \\ &\quad + \sum_{i=k}^{m-1} |f(t_{i+1},y) - f(t_i,y)|.\end{aligned} \tag{30}$$

So, for any partition $P(I) = \{t_1, t_2, \dots, t_m\}$ in $I = [x_1,x_2]$, and for every $y \in J = [y_1,y_2]$ satisfy

$$V_{[x_1,x_2]}(f(.,y)) \le V_{[x_1,c]}\big(f(.,y)\big) + V_{[c,x_2]}\big(f(.,y)\big). \tag{31}$$

Furthermore we will be shown $V_{[x_1,c]}(f(.,y)) + V_{[c,x_2]}(f(.,y)) \le V_I(f(.,y))$.

Taken any partition $P_1(I) = \{t_1, t_2, \dots, t_k\}$ in $[x_1,c]$ and the partition $P_2(I) = \{r_1, r_2, \dots, r_l\}$ in $[c,x_2]$, then $P_1(I) \cup P_2(I) = \{t_1, t_2, \dots, t_k, r_1, r_2, \dots, r_l\}$ where $t_k, r_1 = c$. So, $P_1(I) \cup P_2(I) = \{x_1 = t_1, t_2, \dots, t_k = r_1 = c, r_2, \dots, r_l = x_2\}$ be a partition in $I = [x_1,x_2]$ and apply that

$$\begin{aligned}\sum_{i=1}^{k-1} |\Delta_{10} f(t_{i+1},y)| + \sum_{i=1}^{l-1} |\Delta_{10} f(r_{i+1},y)| &= \sum_{i=1}^{k-1} |f(t_{i+1},y) - f(t_i,y)| + \sum_{i=1}^{l-1} |f(r_{i+1},y) - f(r_i,y)| \\ &= \sum_{i=1}^{k+l-2} |f(t_{i+1},y) - f(t_i,y)| \\ &\le V_I\big(f(.,y)\big).\end{aligned} \tag{32}$$

Therefore for any partition $P_1(I) = \{t_1, t_2, \dots, t_k\}$ on $[x_1,c]$ and $P_2(I) = \{r_1, r_2, \dots, r_l\}$ be partition on $[c,x_2]$ satisfy that

$$\sum_{i=1}^{k-1}|\Delta_{10}f(t_{i+1},y)|+\sum_{i=1}^{l-1}|\Delta_{10}f(r_{i+1},y)|\le V_I(f(.,y)). \tag{33}$$

then

$$V_{[x_1,c]}(f(.,y))+V_{[c,x_2]}(f(.,y))\le V_{[x_1,x_2]}(f(.,y)), \tag{34}$$

So we get $V_I(f(.,y))\le V_{[x_1,c]}(f(.,y))+V_{[c,x_2]}(f(.,y))$ and $V_{[x_1,c]}(f(.,y))+V_{[c,x_2]}(f(.,y))\le V_I(f(.,y))$, then $V_I(f(.,y))=V_{[x_1,c]}(f(.,y))+V_{[c,x_2]}(f(.,y))$.

Because $V_{[x_1,c]}(f(.,y))<\infty$ and $V_{[c,x_2]}(f(.,y))<\infty$ then $V_I(f(.,y))=V_{[x_1,c]}(f(.,y))+V_{[c,x_2]}(f(.,y))<\infty$, so $f(.,y)$ be bounded variation in $I=[x_1,x_2]$ for any $y\in[y_1,y_2]$.

Theorem 3.4 *Let the function $f:I_a^b\to\mathbb{R}$, $x\in I$ be fixed and $f(x,.):J=[y_1,y_2]\to\mathbb{R}$, let $d\in[y_1,y_2]$. If $f(x,.)$ bounded variation on $[y_1,d]$ and $[d,y_2]$, then $f(x,.)$ is bounded variation on $[y_1,y_2]$ and*

$$V_J(f(x,.))=V_{[y_1,d]}(f(x,.))+V_{[d,y_2]}(f(x,.)). \tag{35}$$

Proof. The proof of this theorem similarly with the theorem .

Theorem 3.5 *Given a function $f:I_a^b\to\mathbb{R}$, let $(c,d)\in I_a^b=[x_1,x_2]\times[y_1,y_2]$. If the function f be bounded variation on $I_1=[x_1,c]\times[y_1,d]$, $I_2=[x_1,c]\times[d,y_2]$, $I_3=[c,x_2]\times[y_1,d]$ and $I_4=[c,x_2]\times[d,y_2]$, then the function f be bounded variation on I_a^b and*

$$V_{I_a^b}(f)=V_{I_1}(f)+V_{I_2}(f)+V_{I_3}(f)+V_{I_4}(f). \tag{36}$$

Proof. Taken any partition $\{[t_i,t_{i+1}]\times[s_j,s_{j+1}]\,|\,i=1,\dots,m \text{ and } j=1,\dots,n\}$ in $I_a^b=[x_1,x_2]\times[y_1,y_2]$, let $(c,d)\in I_a^b=[x_1,x_2]\times[y_1,y_2]$ and $(t_{k-1},s_{w-1})<(c,d)<(s_w,t_k)$, for any $k\in\{1,2,\dots,m\}$ and $w\in\{1,2,\dots,n\}$, such that

$$\begin{aligned}
\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}|\Delta_{11}f(t_{i+1},s_{j+1})| &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right| \\
&= \sum_{i=1}^{k-2}\sum_{j=1}^{w-2}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right| \\
&\quad +\left|\left(f(t_k,s_w)-f(t_k,s_{w-1})\right)-\left(f(t_{k-1},s_w)-f(t_{k-1},s_{w-1})\right)\right| \\
&\quad +\sum_{i=k}^{m-1}\sum_{j=w}^{n-1}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right| \\
&= \sum_{i=1}^{k-2}\sum_{j=1}^{w-2}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right| \\
&\quad +\Big|\left(f(c,d)-f(c,s_{w-1})\right)-\left(f(t_{k-1},d)-f(t_{k-1},s_{w-1})\right)+\left(f(c,s_w)-f(c,d)\right)-\left(f(t_{k-1},s_w)-f(t_{k-1},d)\right) \\
&\quad +\left(f(t_k,d)-f(t_k,s_{w-1})\right)-\left(f(c,d)-f(c,s_{w-1})\right)+\left(f(t_k,s_w)-f(t_k,d)\right) \\
&\quad -\left(f(c,s_w)-f(c,d)\right)\Big| \\
&\quad +\sum_{i=k}^{m-1}\sum_{j=w}^{n-1}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right|
\end{aligned} \tag{37}$$

Therefore $(t_{k-1},s_{w-1})<(c,d)<(s_w,t_k)$, then

$$\begin{aligned}
&\le \sum_{i=1}^{k-2}\sum_{j=1}^{w-2}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right| \\
&\quad +\left|\left(f(c,d)-f(c,s_{w-1})\right)-\left(f(t_{k-1},d)-f(t_{k-1},s_{w-1})\right)\right| \\
&\quad +\left|\left(f(c,s_w)-f(c,d)\right)-\left(f(t_{k-1},s_w)-f(t_{k-1},d)\right)\right| \\
&\quad +\left|\left(f(t_k,d)-f(t_k,s_{w-1})\right)-\left(f(c,d)-f(c,s_{w-1})\right)\right| \\
&\quad +\left|\left(f(t_k,s_w)-f(t_k,d)\right)-\left(f(c,s_w)-f(c,d)\right)\right| \\
&\quad +\sum_{i=k}^{m-1}\sum_{j=w}^{n-1}\left|\left(f(t_{i+1},s_{j+1})-f(t_{i+1},s_j)\right)-\left(f(t_i,s_{j+1})-f(t_i,s_j)\right)\right|
\end{aligned} \tag{38}$$

Because the partition

$Q_1=\{t_1,t_2,\dots t_{k-1},c\}\times\{s_1,s_2,\dots,s_{w-1},d\}$ lies at interval $I_1=[x_1,c]\times[y_1,d]$,

$Q_2 = \{t_1, t_2, \dots t_{k-1}, c\} \times \{d, s_w, s_{w+1}, \dots, s_n\}$ lies at interval $I_2 = [x_1, c] \times [d, y_2]$,
$Q_3 = \{c, t_{k+1}, t_{k+2}, \dots t_m\} \times \{s_1, s_2, \dots, s_{w-1}, d\}$ lies at interval $I_3 = [c, x_2] \times [y_1, d]$,
$Q_4 = \{c, t_{k+1}, t_{k+2}, \dots t_m\} \times \{d, s_w, s_{w+1}, \dots, s_n\}$ lies at interval $I_4 = [c, x_2] \times [d, y_2]$.

Then,

$$\begin{aligned}
&\leq \sum_{i=1}^{k-2} \sum_{j=1}^{w-2} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right| \\
&\quad + \left| \left(f(c, d) - f(c, s_{w-1}) \right) - \left(f(t_{k-1}, d) - f(t_{k-1}, s_{w-1}) \right) \right| \\
&\quad + \left| \left(f(c, s_w) - f(c, d) \right) - \left(f(t_{k-1}, s_w) - f(t_{k-1}, d) \right) \right| \\
&\quad + \sum_{i=1}^{k-2} \sum_{j=w+1}^{n-2} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right| \\
&\quad + \sum_{i=k}^{m-1} \sum_{j=1}^{w-2} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right| \\
&\quad + \left| \left(f(t_k, d) - f(t_k, s_{w-1}) \right) - \left(f(c, d) - f(c, s_{w-1}) \right) \right| \\
&\quad + \left| \left(f(t_k, s_w) - f(t_k, d) \right) - \left(f(c, s_w) - f(c, d) \right) \right| \\
&\quad + \sum_{i=k}^{m-1} \sum_{j=w}^{n-1} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right|
\end{aligned} \tag{39}$$

So, it is for arbitrary partitions

$Q = \{[t_i, t_{i+1}] \times [s_j, s_{j+1}] | i = 1, \dots, m \text{ dan } j = 1, \dots, n\}$ in $I_a^b = [x_1, x_2] \times [y_1, y_2]$, satisfy $V_{I_a^b}(f) \leq V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f)$.

Next, will be shown that $V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f) \leq V_{I_a^b}(f)$.

Taken any partition $Q_1 = \{t_1, t_2, \dots t_k\} \times \{s_1, s_2, \dots, s_w\}$ on $I_1 = [x_1, c] \times [y_1, d]$, partition $Q_2 = \{t_1, t_2, \dots t_k\} \times \{u_1, u_2, \dots, u_q\}$ on $I_2 = [x_1, c] \times [d, y_2]$, partition $Q_3 = \{r_1, r_2, \dots, r_l\} \times \{s_1, s_2, \dots, s_w\}$ on $I_3 = [c, x_2] \times [y_1, d]$, and partition $Q_3 = \{r_1, r_2, \dots, r_l\} \times \{u_1, u_2, \dots, u_q\}$ on $I_4 = [c, x_2] \times [d, y_2]$, then

$Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \{t_1, t_2, \dots t_k, r_1, r_2, \dots, r_l\} \times \{s_1, s_2, \dots, s_w, u_1, u_2, \dots, u_q\}$, where $t_k, r_1 = c$ and $s_w, u_1 = d$. Such that

$$Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \{t_1, t_2, \dots t_k = r_1 = c, r_2, \dots, r_l\} \times \{s_1, s_2, \dots, s_w = u_1 = d, u_2, \dots, u_q\}$$

be partition in $I_a^b = [x_1, x_2] \times [y_1, y_2]$ and apply that

$$\begin{aligned}
&\sum_{i=1}^{k-1} \sum_{j=1}^{w-1} |\Delta_{11} f(t_{i+1}, s_{j+1})| + \sum_{i=1}^{k-1} \sum_{j=1}^{q-1} |\Delta_{11} f(t_{i+1}, u_{j+1})| + \sum_{i=1}^{l-1} \sum_{j=1}^{n-1} |\Delta_{11} f(r_{i+1}, s_{j+1})| + \sum_{i=1}^{l-1} \sum_{j=1}^{q-1} |\Delta_{11} f(r_{i+1}, u_{j+1})| \\
&= \sum_{i=1}^{k-1} \sum_{j=1}^{w-1} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right| \\
&\quad + \sum_{i=1}^{k-1} \sum_{j=1}^{q-1} \left| \left(f(t_{i+1}, u_{j+1}) - f(t_{i+1}, u_j) \right) - \left(f(t_i, u_{j+1}) - f(t_i, u_j) \right) \right| \\
&\quad + \sum_{i=1}^{l-1} \sum_{j=1}^{n-1} \left| \left(f(r_{i+1}, s_{j+1}) - f(r_{i+1}, s_j) \right) - \left(f(r_i, s_{j+1}) - f(r_i, s_j) \right) \right| \\
&\quad + \sum_{i=1}^{l-1} \sum_{j=1}^{q-1} \left| \left(f(r_{i+1}, u_{j+1}) - f(r_{i+1}, u_j) \right) - \left(f(r_i, u_{j+1}) - f(r_i, u_j) \right) \right| \\
&= \sum_{i=1}^{2k+2l-4} \sum_{j=1}^{2w+2q-4} \left| \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j) \right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j) \right) \right|.
\end{aligned} \tag{40}$$

Therefore for any partition Q_1 on I_1, partition Q_2 on I_2, partition Q_3 on I_3, and partition Q_4 on I_4, satisfy

$$V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f) \leq V_{I_a^b}(f). \quad (41)$$

Because that we get $V_{I_a^b}(f) \leq V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f)$ and $V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f) \leq V_{I_a^b}(f)$, then

$$V_{I_a^b}(f) = V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f). \quad (42)$$

Notice that $V_{I_1}(f) < \infty$, $V_{I_2}(f) < \infty$, $V_{I_3}(f) < \infty$, and $V_{I_4}(f) < \infty$ so we can conclude that $V_{I_a^b}(f) = V_{I_1}(f) + V_{I_2}(f) + V_{I_3}(f) + V_{I_4}(f) < \infty$, so f be bounded variation on I_a^b.

CONCLUSION

Function of bounded variation of space $\mathbb{R}^2$ is the development of function of bounded variation to space $\mathbb{R}$. The concepts developed are partition, definition and properties of function of bounded variation. The proof regarding the properties of function of bounded variation in space $\mathbb{R}^2$ is carried out in the same way as in space $\mathbb{R}$. Every of function of bounded variation is a finite function. This study also discusses the relationship between a function of bounded variation and a monotonic function, an increasing monotonic function can be expressed as the difference from the end point of the interval. A function of bounded variation of two variables is also a finite function on each subset of the $\mathbb{R}^2$. Each variation of the function of bounded variation can be expressed as the sum of each of its sub-squares.

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