The Distance between Two Convex Sets in Hilbert Space

by Susilo Hariyanto

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Abstract. In this paper we will discuss how to determine the distance between two convex sets in Hilbert space. This problem came from measuring the shortest distance between two cities, which consider by determining the distance between two buildings in each city. In this problem, the cities are considered as the sets and the buildings are points. Furthermore, based on this problem, it is generalized to determine the distance between two convex sets in Hilbert space that solved by optimization concept by measuring maximal distance between two parallel supporting hyperplanes that separate them. Therefore, it is given some example to understanding, such as the distance between two normed balls, ellipsoids, and linear varieties.

INTRODUCTION

Let $\mathcal{Y}$ be a city and let $\boldsymbol{z}$ be some object outside $\mathcal{Y}$. City $\mathcal{Y}$ has a lot of buildings, such as markets, schools, hospitals, government offices, and others. We can measure the distance between the city $\mathcal{Y}$ and $\boldsymbol{z}$ by determining the nearest building(s) in city $\mathcal{Y}$ to $\boldsymbol{z}$. This problem is called as best approximation problem. The related theories in this problem are Approximation Theory, Functional Analysis, Convex Analysis, Optimization, Linear Algebra, and others. If the city is declared as the set, the buildings are the points, and $\boldsymbol{z} \in \mathcal{Z}$, we will compute the distance between the sets $\mathcal{Y}$ and $\mathcal{Z}$ by finding the nearest point between them, such that $\|\boldsymbol{y} - \boldsymbol{z}\|$ attains the minimum value, where $\boldsymbol{y} \in \mathcal{Y}$ and $\boldsymbol{z} \in \mathcal{Z}$.

Best approximation problem was introduced in ref. [1] who proposed Minimum Norm Duality (MND) theorem to determine the distance between a convex set and a point outside it. This theorem said that the shortest distance between a convex set $\mathcal{Y}$ and a point $\boldsymbol{z} \notin \mathcal{Y}$ is equal to the maximum distance between a point $\boldsymbol{z} \notin \mathcal{Y}$ to any separating hyperplane of $\boldsymbol{z}$ and $\mathcal{Y}$ (see fig. FIGURE **2**). Next, ref. [2] computed the distance between two convex polygons by computational complexity. Furthermore, ref. [3] studied minimum distance to the complement of a convex set, and ref. [4] modified MND theorem that proposed by [1] to compute the distance between two convex sets.

Based on these studies, it is shown how to compute the distance between two sets by measuring the distance between two parallel hyperplanes that separate them. Some restriction that applied in this paper is

1. The solution of best approximation is unique, so the sets must be convex, [5,6]. Since we compute the minimum distance by measuring the distance between two parallel hyperplane that separate them, the existence of separating hyperplane cannot be guaranteed if one of the set is not convex, see fig. FIGURE **1**.
2. Some of studies [2–4,7] have considered the minimum distance problem in Euclidean space $\mathbb{R}^n$, although Luenberger [1] and Deutsch [5] has defined this problem in inner product space (or Hilbert space). Based on fact that hyperplane is defined on inner product space, we will consider this problem in larger space, that Hilbert space.

The plan of this paper is as follows. In section 1 contains necessary background. Section 2 explains the basic facts on norms. In section 3, it is shown some assertions of hyperplane, and section 4 shows the theories about the best approximation, and we give some examples in section 5.

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NORMS

As a basic of minimum distance, first we show the definition of norm in definition 1 below.

Definition 1 [6] *Norm* $\|\cdot\|$ *is a mapping from linear space to non-negative real number such that the following properties are satisfied:*

1. $\|\boldsymbol{u}\| \geq 0$ (positivity)
2. $\|\alpha\boldsymbol{u}\| = |\alpha|\|\boldsymbol{u}\|$ (homogeneity)
3. $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (triangle inequality)

For all $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in the linear space, and $\alpha \in \mathbb{R}$.

Note that norm is a convex function. The dual norm of $\|\cdot\|$ is denoted by $\|\cdot\|'$, that obtained by this following way. Given a vector $\boldsymbol{w}$ in a linear space, then $\|\boldsymbol{w}\|' = \max_{\|\boldsymbol{u}\| \leq 1} \langle \boldsymbol{u}, \boldsymbol{w} \rangle$, and the Hölder inequality states that $|\langle \boldsymbol{u}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{u}\|\|\boldsymbol{w}\|'$. If $|\langle \boldsymbol{u}, \boldsymbol{w} \rangle| = \|\boldsymbol{u}\|\|\boldsymbol{w}\|'$ then it is aligned with respect to $\|\cdot\|$ or $\|\cdot\|'$.

One example of dual norms is related to the $L_p[a,b]$ norm, which contains all integrated function in $[a,b]$ that defined as

$$\|u\|_p = \left(\int_a^b |u(t)|^p dt \right)^{\frac{1}{p}}$$

Where $1 < p < \infty$. The dual norm is $L_q[a,b]$ norm

$$\|w\|_q = \left(\int_a^b |w(t)|^q dt \right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, the dual norm is itself, and L_2-space is contained in Hilbert space.

HYPERPLANE

According to ref. [8,9], hyperplane that denoted by $\mathcal{H}$, is defined in inner product space or Hilbert space as a set of vectors $\boldsymbol{x}$ which satisfies $\mathcal{H} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}$. Hyperplane divide a linear space into two half spaces such that

$$\mathcal{H}^{\leq} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq \alpha\}$$
$$\mathcal{H}^{\geq} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq \alpha\},$$

If the boundary line is excluded, then we have the two open half spaces

$$\mathcal{H}^{<} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle < \alpha\}$$
$$\mathcal{H}^{>} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle > \alpha\}.$$

In other words, half spaces $\mathcal{H}^{<}$ and $\mathcal{H}^{>}$ are separated by hyperplane $\mathcal{H}$, or $\mathcal{H}$ is a called as **separating hyperplane**. This below theorem guarantees the existence of separating hyperplane.

Theorem 2 [10] (Separating Hyperplane Theorem) *Let* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be nonempty disjoint convex sets. Then there exist* $\boldsymbol{a} \neq \boldsymbol{0}$ *and* $\alpha \in \mathbb{R}$ *such that* $\langle \boldsymbol{a}, \boldsymbol{x} \rangle - \alpha \leq 0, \forall x \in \mathcal{Y}$ *and* $\langle \boldsymbol{a}, \boldsymbol{x} \rangle - \alpha \geq 0, \forall \boldsymbol{x} \in \mathcal{Z}$.

It is equivalent to say that separating hyperplane $\mathcal{H}$ always exist if both $\mathcal{Y}$ and $\mathcal{Z}$ are convex sets. But if either $\mathcal{Y}$ or $\mathcal{Z}$ is not convex, theorem 2 fails, as shown on fig. 1.

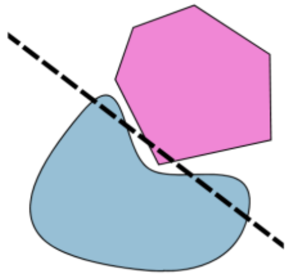

FIGURE 1. Separating theorem does not apply if one of the bodies is not convex

If hyperplane $\mathcal{H}$ is tangent to a set, then $\mathcal{H}$ is **supporting hyperplane**, as shown on definition 3.

Definition 3 [4] *Let $\mathcal{Y}$ be a convex set. Hyperplane $\mathcal{H} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}$ is said to be a supporting hyperplane of $\mathcal{Y}$ if*

$$\sup_{\boldsymbol{x} \in \mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha.$$

Since $x \in \mathcal{H}$, definition 3 also means that x and convex set $\mathcal{Y}$ are separated by hyperplane $\mathcal{H}$, and hyperplane $\mathcal{H}$ supports $\mathcal{Y}$ at $\boldsymbol{x}^*$. The first step to measure the distance between two convex sets is choosing a pair of parallel separating hyperplanes that support each convex sets. In the next section, it is shown how to compute the distance between two convex sets through the dual problem.

THE DISTANCE BETWEEN TWO CONVEX SETS

Let $\|\cdot\|$ be some arbitrary norm and $\|\cdot\|'$ the corresponding dual norm. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two nonempty disjoint convex sets. The distance between two convex sets $\mathcal{Y}$ and $\mathcal{Z}$ is defined as

$$\operatorname{dist}(\mathcal{Y}, \mathcal{Z}) = \inf\{\|\boldsymbol{y} - \boldsymbol{z}\| \,|\, \boldsymbol{y} \in \mathcal{Y}, \boldsymbol{z} \in \mathcal{Z}\} \tag{1}$$

According to ref. [11] and [10], hyperplane is an example of convex sets. In $\mathbb{R}^n$ space, hyperplane can be visualized as flat. To compute the distance between two convex sets by its dual, we must construct the supporting hyperplanes. So, in this section we will proof that the minimum distance between two convex sets is equal to the maximum distance between two separating hyperplanes.

First, we consider the distance between two parallel hyperplanes. Let $\mathcal{H} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}$ be a hyperplane, and $\boldsymbol{z} \notin \mathcal{H}$. By equation (1), the distance between $\boldsymbol{z}$ and $\mathcal{H}$ is

$$\operatorname{dist}(\mathbf{z}, \mathcal{H}) = \inf_{\mathbf{x} \in \mathcal{H}} \|\boldsymbol{z} - \boldsymbol{x}\|.$$

Ref. [7] has proved the distance between point and hyperplane is

$$\operatorname{dist}(\mathbf{z}, \mathcal{H}) = \left| \frac{\langle \boldsymbol{a}, \boldsymbol{z} \rangle - \alpha}{\|\boldsymbol{a}\|'} \right| \tag{2}$$

Since $\boldsymbol{z} \notin \mathcal{H}$, we must choose two disjoint hyperplanes $\mathcal{H}_1 = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha_1\}$ and $\mathcal{H}_2 = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha_2\}$. The distance between them is defined as

$$\operatorname{dist}(\mathcal{H}_1, \mathcal{H}_2) = \inf\{\|\boldsymbol{x_1} - \boldsymbol{x_2}\| \,|\, \boldsymbol{x_1} \in \mathcal{H}_1, \boldsymbol{x_2} \in \mathcal{H}_2\} \tag{3}$$

By (2), we obtain

$$\operatorname{dist}(\mathcal{H}_1, \mathcal{H}_2) = \left| \frac{\langle \boldsymbol{a}, \boldsymbol{x_1} \rangle - \alpha_2}{\|\boldsymbol{a}\|'} \right|, \qquad \forall \boldsymbol{x_1} \in \mathcal{H}_1$$

Therefore, since $\langle \boldsymbol{a}, \boldsymbol{x_1} \rangle = \alpha_1$, $\forall \boldsymbol{x_1} \in \mathcal{H}_1$,

$$\operatorname{dist}(\mathcal{H}_1, \mathcal{H}_2) = \left| \frac{\alpha_1 - \alpha_2}{\|\boldsymbol{a}\|'} \right| \tag{4}$$

Look again eq. (1). Let $\mathcal{Y} - \mathcal{Z} = \mathcal{X}$, and $\mathbf{0} \notin \mathcal{X}$. The distance between $\mathcal{Y}$ and $\mathcal{Z}$ is equal to

$$\operatorname{dist}(\mathcal{Y}, \mathcal{Z}) = \operatorname{dist}(\mathbf{0}, \mathcal{X}) = \inf_{\boldsymbol{x} \in \mathcal{X}} \|\boldsymbol{x}\|$$

Since $\mathcal{Y}$ and $\mathcal{Z}$ are convex, $\mathcal{X}$ is also convex.

Lemma 4 [4] (The Least Norm Problem) *There exists a point* $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ *such that* $\|\boldsymbol{x}^*\| \leq \|\boldsymbol{x}\|, \forall \boldsymbol{x} \in \boldsymbol{\mathcal{X}}$.

Lemma 4 expresses an optimization problem

$$\begin{aligned} &\text{minimize } \ \|\boldsymbol{x}\| \\ &\text{subject to } \boldsymbol{x} \in \mathcal{X} \end{aligned} \tag{5}$$

is always solvable, that attains a minimizer $\boldsymbol{x}^* \in \mathcal{X}$. But, if $\|\cdot\|$ is a strictly convex norm, $\boldsymbol{x}^*$ is unique.

Now, let's construct the dual problem of (5). For this purpose, we show the support functions

$$\begin{aligned} \alpha(\boldsymbol{a}) &= \sup_{\boldsymbol{y} \in \mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{y} \rangle; \\ \beta(\boldsymbol{a}) &= \inf_{\boldsymbol{z} \in \mathcal{Z}} \langle \boldsymbol{a}, \boldsymbol{z} \rangle \end{aligned} \tag{6}$$

which are well defined for any vector $\boldsymbol{a}$. If any $\boldsymbol{a} \neq \boldsymbol{0}$ and $\alpha(\boldsymbol{a}) \leq \beta(\boldsymbol{a})$ then the parallel hyperplanes

$$\mathcal{H}_\alpha = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha(\boldsymbol{a})\},\ \mathcal{H}_\beta = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \beta(\boldsymbol{a})\}$$

Separates $\mathcal{Y}$ and $\mathcal{Z}$. By (4),

$$\operatorname{dist}(\mathcal{H}_\alpha, \mathcal{H}_\beta) = \frac{\beta(\boldsymbol{a}) - \alpha(\boldsymbol{a})}{\|\boldsymbol{a}\|'}.$$

In figure below, it is shown how to choose the hyperplane for calculating the distance of two convex sets.

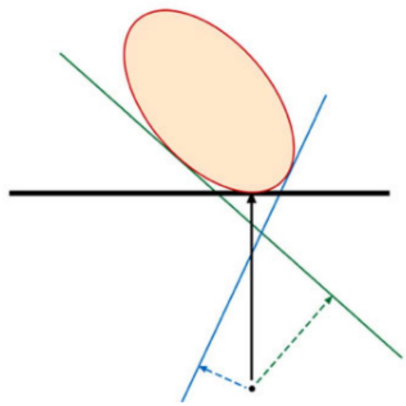

FIGURE 2. Hyperplane choosing

Fig. 2 shows that the lines (hyperplanes) separate the point and the set. To compute the distance, we must choose a point $\boldsymbol{x} \in \mathcal{Y}$ that closest to $\boldsymbol{z} \notin \mathcal{Y}$. The closest point in set $\mathcal{Y}$ is supported by the black hyperplane, which make a maximal separation between the set and the point. Since the hyperplane equation is $\mathcal{H} = \{\boldsymbol{x} | \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}$, we must find the vector $\boldsymbol{a}$ such that $\operatorname{dist}(\boldsymbol{z}, \mathcal{H})$ attains the maximal value. Ref. [4] proposed this theorem as the dual of equation (5).

Theorem 5 [4] *Let* $\boldsymbol{\mathcal{A}}$ *denote the set of all points* $\boldsymbol{a}$ *such that hyperplanes* $\boldsymbol{\mathcal{H}_\alpha}$ *and* $\boldsymbol{\mathcal{H}_\beta}$ *separate* $\boldsymbol{\mathcal{Y}}$ *and* $\boldsymbol{\mathcal{Z}}$*. The formula of maximal separation problem is*

$$\begin{aligned} &\text{maximize } \ \sigma(\boldsymbol{a}) = \beta(\boldsymbol{a}) - \alpha(\boldsymbol{a}) \\ &\text{subject to } \boldsymbol{a} \in \mathcal{A}. \end{aligned} \tag{7}$$

Let $\boldsymbol{a}^*$be solution of this problem, then $\|\boldsymbol{a}^*\|' = 1$, and $\sigma(\boldsymbol{a}^*) = \frac{\beta(\boldsymbol{a}) - \alpha(\boldsymbol{a})}{\|\boldsymbol{a}\|'}$.

Proof. Let $\mathcal{H}_\alpha = \{\boldsymbol{x_1} | \langle \boldsymbol{a}, \boldsymbol{x_1} \rangle = \alpha(\boldsymbol{a})\}$ and $\mathcal{H}_\beta = \{\boldsymbol{x_2} | \langle \boldsymbol{a}, \boldsymbol{x_2} \rangle = \beta(\boldsymbol{a})\}$. It is shown the distance

$$dist(\mathcal{H}_1, \mathcal{H}_2) = \|\boldsymbol{x_1} - \boldsymbol{x_2}\|$$

If $\boldsymbol{a}$ solve (7), then

$$\begin{aligned} \sigma(\boldsymbol{a}) &= \beta(\boldsymbol{a}) - \alpha(\boldsymbol{a}) \\ &= \langle \boldsymbol{a}, \boldsymbol{x_2} \rangle - \langle \boldsymbol{a}, \boldsymbol{x_1} \rangle \\ &= \langle \boldsymbol{a}, \boldsymbol{x_1} - \boldsymbol{x_2} \rangle \\ &= \|\boldsymbol{a}\|' \|\boldsymbol{x_1} - \boldsymbol{x_2}\| \end{aligned}$$

Since $\sigma(\boldsymbol{a}) = \|\boldsymbol{a}\|'\|\boldsymbol{x_1} - \boldsymbol{x_2}\|$, and $dist(\mathcal{H}_1,\mathcal{H}_2) = \|\boldsymbol{x_1} - \boldsymbol{x_2}\|$, then $\sigma(\boldsymbol{a}) = dist(\mathcal{H}_1,\mathcal{H}_2)$ is attained when $\|\boldsymbol{a}\|' = 1$ or $\frac{\beta(\boldsymbol{a})-\alpha(\boldsymbol{a})}{\|\boldsymbol{a}\|'}$.

In practice, it is convenient to write eq. (7) in the form

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = \inf_{\boldsymbol{z} \in \mathcal{Z}} \langle \boldsymbol{a}, \boldsymbol{z} \rangle - \sup_{\boldsymbol{y} \in \mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{y} \rangle \\ &\text{subject to} \|\boldsymbol{a}\|' \leq 1 \end{aligned} \tag{8}$$

The justification for replacing (7) and (8) lies in the definition of $\mathcal{A}$. Recall that $\boldsymbol{a} \in \mathcal{A}$ iff $\sigma(\boldsymbol{a}) \geq 0$. Consequently any solution of (8) solves (7) and vice versa. Ref. [4] gave this theorem for summarizing this result.

Theorem 6 (Dax' Minimum Norm Duality Theorem). *The dual of the least norm problem* (5) *is the maximum separation problem* (8) *and both problems are solvable. Let* $\mathbf{x}^* \in \boldsymbol{X}$ *solve* (5) *and* let $\mathbf{a}^*$ *solve the* (8), *then*

$$\|\mathbf{a}^*\|' = 1 \textit{ and } \sigma(\mathbf{a}^*) = \|\mathbf{x}^*\| = \mathrm{dist}(\mathcal{Y},\mathcal{Z})$$

Furthermore, if $\mathbf{x}^* \neq \mathbf{0}$ then $\mathbf{x}^*$ and $-\mathbf{a}^*$ are aligned. That is,

$$\langle -\mathbf{a}^*, \mathbf{x}^* \rangle = \|-\mathbf{a}^*\|'\|\mathbf{x}^*\| = \|\mathbf{x}^*\|$$

SOME EXAMPLES

In this section, it is given some examples to enable us to understand the concept of minimum distance. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two disjoint convex closed sets in Hilbert space. The primal problem to be solved has the form

$$\begin{aligned} &\text{minimize} \|\boldsymbol{y} - \boldsymbol{z}\| \\ &\text{minimize } \boldsymbol{y} \in \mathcal{Y}, \boldsymbol{z} \in \mathcal{Z} \end{aligned} \tag{9}$$

The least distance problems consider some kinds of convex sets, such as normed balls, normed ellipsoids, and polytopes. The study of problems illustrate how the general definition of the dual problem,

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = \inf_{\boldsymbol{z} \in \mathcal{Z}} \langle \boldsymbol{a}, \boldsymbol{z} \rangle - \sup_{\boldsymbol{y} \in \mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{y} \rangle \\ &\text{subject to} \|\boldsymbol{a}\|' \leq 1 \end{aligned} \tag{10}$$

is casted into a specific maximization problem when $\mathcal{Y}$ and $\mathcal{Z}$ are specified.

The examples start by considering two disjoint convex sets, where both $\mathcal{Y}$ and $\mathcal{Z}$ are not singleton. The original problem (9) can be formulated as a best approximation problem to find the shortest distance between two convex sets by computing the distance two nearest points in each set. This gives the problem a second geometric interpretation and enables us to apply the problem (10). Consider for example the distance between two linear varieties, two normed balls, and ellipsoids.

The Distance between Two Linear Varieties

Ref. [12] said that linear variety is obtained by translating linear subspace. The standard form of linear varieties is

$$\mathcal{Y} = \{\tilde{\boldsymbol{y}} + B\boldsymbol{u}\}, \qquad \mathcal{Z} = \{\tilde{\boldsymbol{z}} - C\boldsymbol{v}\}$$

Where $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{z}}$ are the vector in Hilbert space, which act as translation factors, B and C are the operators, and $B\boldsymbol{u}$ and $C\boldsymbol{v}$ are the linear subspaces. The minimum distance between $\mathcal{Y}$ and $\mathcal{Z}$ has the form

$$\text{minimize } \|\tilde{\boldsymbol{y}} + B\boldsymbol{u} - \tilde{\boldsymbol{z}} + C\boldsymbol{v}\|$$

or, simply,

$$\text{minimize } \|\boldsymbol{b} - A\boldsymbol{x}\| \tag{11}$$

where $\boldsymbol{b} = \boldsymbol{z} - \boldsymbol{y}, A = [B, C]$ and $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{v})$ denotes the unknown vector. The problem (11) has a new geometric interpretation, which is to seek the shortest distance between a point $\boldsymbol{b}$ and the subspace

$$\mathcal{S} = \{A\boldsymbol{x}\} = Range(A)$$

To find the distance between $\boldsymbol{b}$ and $\mathcal{S}$, ref. [1] proposed this theorem.

Theorem 7 [1] *Let* $\boldsymbol{b}$ *be an element in a real normed linear space* $\boldsymbol{X}$ *and let* $\mathbf{d}$ *denote its distance from the subspace* $\boldsymbol{S}$. *Then,*

$$d = \inf_{s \in \mathcal{S}} \|\boldsymbol{b} - \boldsymbol{s}\| = \max_{\substack{\|\boldsymbol{a}^*\| \leq 1 \\ \boldsymbol{a}^* \in \mathcal{S}^\perp}} \langle \boldsymbol{b}, \boldsymbol{a} \rangle \tag{12}$$

Where the maximum on the right side is attained for $\boldsymbol{a}^* \in \mathcal{S}^\perp$, *that is the orthogonal complement of* $\mathcal{S}$.

Proof. This theorem will be proven from the left side to the right side. In problem (12), it is said that $\boldsymbol{s} \in \mathcal{S}$, then $\boldsymbol{s} = A\boldsymbol{x}$. Let $\boldsymbol{w} \in \mathcal{S}^{\perp}$, such that

$$\begin{aligned}\mathcal{S}^{\perp} &= \{\boldsymbol{w} | \langle A\boldsymbol{x}, \boldsymbol{w}\rangle = 0\}\\ &= \{\boldsymbol{w} | \langle \boldsymbol{x}, A^{\dagger}\boldsymbol{w}\rangle = 0\}\\ &= \{\boldsymbol{w} | A^{\dagger}\boldsymbol{w} = \boldsymbol{0}\} = Null(A^{\dagger})\end{aligned}$$

If we apply the formula (8), then $\sigma(\boldsymbol{a}^*) \geq 0$. So, $\inf\langle \boldsymbol{b}, \boldsymbol{a}\rangle \geq \sup\langle A\boldsymbol{x}, \boldsymbol{a}\rangle$. Now, let us consider the dual problem. We have two values of $\sup_{\boldsymbol{s} \in \mathcal{S}} \langle \boldsymbol{a}, \boldsymbol{s}\rangle$, that is

$$\sup_{\boldsymbol{s} \in \mathcal{S}} \langle \boldsymbol{a}, \boldsymbol{s}\rangle = \begin{cases} 0, & \boldsymbol{a} \in \mathcal{S}^{\perp} \\ \infty, & otherwise \end{cases}$$

Since $\langle \boldsymbol{b}, \boldsymbol{a}\rangle$ is finite, then $\boldsymbol{a}$ must be belong to the orthogonal complement of $\mathcal{S}$.

Hence by formula (8), the dual of (11) has the form

$$\begin{aligned}&\text{maximize } \langle \boldsymbol{b}, \boldsymbol{a}\rangle\\ &\text{subject to } A^{\dagger}\boldsymbol{a} = \boldsymbol{0}, \|\boldsymbol{a}\|' \leq 1\end{aligned} \qquad (13)$$

The Distance between Two Normed Balls

Next, we consider the distance between two normed balls in standard form

$$\mathcal{Y} = \{\boldsymbol{y_c} + \boldsymbol{y} | \|\boldsymbol{y}\| \leq \rho_1\}, \text{ and } \mathcal{Z} = \{\boldsymbol{z_c} - \boldsymbol{z} | \|\boldsymbol{z}\| \leq \rho_2\},$$

where $\boldsymbol{y_c}$ and $\boldsymbol{x_c}$ are the center of the balls, and scalar ρ_1 and ρ_2 are the radii. In this case the least distance problem (9) become

$$\begin{aligned}&\text{minimize} \|\boldsymbol{y_c} - \boldsymbol{z_c} + \boldsymbol{y} + \boldsymbol{z}\|\\ &\text{subject to} \|\boldsymbol{y}\| \leq 1, \|\boldsymbol{z}\| \leq 1\end{aligned} \qquad (14)$$

To construct the dual problem, it is necessary to construct two hyperplanes $\alpha(\boldsymbol{a})$ and $\beta(\boldsymbol{a})$ that support $\mathcal{Y}$ and $\mathcal{Z}$ respectively. So,

$$\begin{aligned}\beta(\boldsymbol{a}) &= \inf_{\boldsymbol{z} \in \mathcal{Z}} \langle \boldsymbol{a}, \boldsymbol{z}\rangle = \langle \boldsymbol{a}, \boldsymbol{z_c} - \boldsymbol{z}\rangle\\ &= \langle \boldsymbol{a}, \boldsymbol{z_c}\rangle - \rho_2 \|\boldsymbol{a}\|'\end{aligned}$$

and

$$\begin{aligned}\alpha(\boldsymbol{a}) &= \sup_{\boldsymbol{y} \in \mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{y}\rangle = \langle \boldsymbol{a}, \boldsymbol{y_c} + \boldsymbol{y}\rangle\\ &= \langle \boldsymbol{a}, \boldsymbol{y_c}\rangle + \rho_1 \|\boldsymbol{a}\|'\end{aligned}$$

So, the dual of (14) has the form

$$\begin{aligned}&\text{maximize } \sigma(\boldsymbol{a}) = -\langle \boldsymbol{a}, \boldsymbol{y_c} - \boldsymbol{z_c}\rangle - \rho_1 \|\boldsymbol{a}\|' - \rho_2 \|\boldsymbol{a}\|'\\ &\text{subject to} \|\boldsymbol{a}\|' \leq 1\end{aligned} \qquad (15)$$

To solve problem (15), we use KKT condition $\nabla\sigma + \mu\nabla g = 0$, where σ denotes the objective, μ is the Lagrange multiplier, and g is the constrain. So,

$$\begin{aligned}\nabla\sigma + \mu\nabla g &= 0\\ -(\boldsymbol{y_c} - \boldsymbol{z_c}) + \frac{\rho_1 + \rho_2 - \mu}{\|\boldsymbol{a}\|} &= 0\\ \mu &= (\rho_1 + \rho_2) - \|\boldsymbol{z_c} - \boldsymbol{y_c}\|\end{aligned}$$

Since $\mu = (\rho_1 + \rho_2) - \|\boldsymbol{y_c} - \boldsymbol{x_c}\|$, we obtain that $\boldsymbol{a} = \frac{\boldsymbol{z_c} - \boldsymbol{y_c}}{\|\boldsymbol{z_c} - \boldsymbol{y_c}\|}$, and

$$\begin{aligned}\sigma(\boldsymbol{a}) &= -\langle \frac{\boldsymbol{z_c} - \boldsymbol{y_c}}{\|\boldsymbol{z_c} - \boldsymbol{y_c}\|}, \boldsymbol{z_c} - \boldsymbol{y_c}\rangle - \rho_1 \|\boldsymbol{a}\| - \rho_2 \|\boldsymbol{a}\|\\ &= \|\boldsymbol{z_c} - \boldsymbol{y_c}\| - (\rho_1 + \rho_2)\end{aligned}$$

So, the distance of between two circles is equal to $\sigma(\boldsymbol{a}) = \|\boldsymbol{z_c} - \boldsymbol{y_c}\| - (\rho_1 + \rho_2)$.

Now, let us consider if $\mathcal{Z}$ is a singleton, that is $\mathcal{Z} = \{\hat{\boldsymbol{z}}\}$. Then the case (15) become

$$\begin{aligned}&\text{maximize } \sigma(\boldsymbol{a}) = -\langle \boldsymbol{a}, \boldsymbol{y_c} - \hat{\boldsymbol{z}}\rangle - \rho_1 \|\boldsymbol{a}\|\\ &\text{subject to} \|\boldsymbol{a}\|' \leq 1\end{aligned} \qquad (16)$$

Similarly, we obtain $\mu = \rho_1 - \|\hat{\boldsymbol{z}} - \boldsymbol{y_c}\|$, $\boldsymbol{a} = \frac{\hat{\boldsymbol{z}} - \boldsymbol{y_c}}{\|\hat{\boldsymbol{z}} - \boldsymbol{y_c}\|}$, and the distance is $\sigma(\boldsymbol{a}) = \|\hat{\boldsymbol{z}} - \boldsymbol{y_c}\| - \rho_1$.

The Distance between Two Normed Ellipsoids

According to ref. [4] and [10], ellipsoids are the sets that denoted by

$$\mathcal{Y} = \{\boldsymbol{y_c} + B\boldsymbol{u} | \|\boldsymbol{u}\| \le 1\}, \qquad \mathcal{Z} = \{\boldsymbol{z_c} - C\boldsymbol{v} | \|\boldsymbol{v}\| \le 1\}$$

Where $\boldsymbol{y_c}$ and $\boldsymbol{z_c}$ are the center of ellipsoids, B and C are the operators, and $\|\cdot\|$ denotes the norm. The operator B and C determines how far the ellipsoid extends in every directions from $\boldsymbol{y_c}$, and the length of the semi-axes of $\mathcal{Y}$ and $\mathcal{Z}$ are given by $\sqrt{\kappa_i(B^2)}$ and $\sqrt{\lambda_i(C^2)}$, where $\kappa_i(B^2)$ and $\lambda_i(C^2)$ are the eigenvalues of B^2 and C^2 respectively. If $\mathcal{Y}$ is normed ball, then $B = \rho I$, where I is the identity, and ρ is the radius.

The distance between $\mathcal{Y}$ and $\mathcal{Z}$ is obtained by solving the minimum norm problem

$$\begin{aligned} &\text{minimize } \|\boldsymbol{y_c} - \boldsymbol{z_c} + B\boldsymbol{u} + C\boldsymbol{v}\| \\ &\text{subject to} \|\boldsymbol{u}\| \le 1, \|\boldsymbol{v}\| \le 1 \end{aligned} \tag{17}$$

As before, we can verify that

$$\begin{aligned} \inf_{\boldsymbol{z}\in\mathcal{Z}} \langle \boldsymbol{a}, \boldsymbol{z}\rangle &= \langle \boldsymbol{a}, \boldsymbol{z_c}\rangle + \inf_{\|\boldsymbol{v}\| \le 1} \langle \boldsymbol{a}, -C\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{a}, \boldsymbol{z_c}\rangle - \|C^\dagger \boldsymbol{a}\|' \end{aligned}$$

and

$$\begin{aligned} \sup_{\boldsymbol{y}\in\mathcal{Y}} \langle \boldsymbol{a}, \boldsymbol{y}\rangle &= \langle \boldsymbol{a}, \boldsymbol{y_c}\rangle + \sup_{\|u\| \le 1} \langle \boldsymbol{a}, B\boldsymbol{u}\rangle \\ &= \langle \boldsymbol{a}, \boldsymbol{y_c}\rangle + \|B^\dagger \boldsymbol{a}\|' \end{aligned}$$

Where $B^\dagger$ and $C^\dagger$ denote the adjoint of B and C respectively, but if we put matrix as the operator, the adjoint is its transpose. Then, the dual of (17) has the form

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = -\langle \boldsymbol{a}, \boldsymbol{y_c} - \boldsymbol{z_c}\rangle - \|B^\dagger \boldsymbol{a}\|' - \|C^\dagger \boldsymbol{a}\|' \\ &\text{subject to} \|\boldsymbol{a}\|' \le 1 \end{aligned} \tag{18}$$

Now, consider when $\mathcal{Z}$ turns to singleton, that is $\mathcal{Z} = \{\hat{\boldsymbol{z}}\}$. Then the problem (17) become

$$\begin{aligned} &\text{minimize } \|B\boldsymbol{u} + (\boldsymbol{y_c} - \hat{\boldsymbol{z}})\| \\ &\text{subject to} \|\boldsymbol{u}\| \le 1 \end{aligned} \tag{19}$$

Let $\boldsymbol{g} = \boldsymbol{y_c} - \hat{\boldsymbol{z}}$, then the dual of (19) has the form

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = -\langle \boldsymbol{g}, \boldsymbol{a}\rangle - \|B^\dagger \boldsymbol{a}\|' \\ &\text{subject to} \|\boldsymbol{a}\|' \le 1 \end{aligned} \tag{20}$$

Usually in Euclidean space $\mathbb{R}^n$, the operator that used is symmetric and invertible matrix. Below, it is given an example to compute the distance between two ellipsoids that solved by applying KKT Theorem (in ref.[13], p.398). Let $\mathcal{Y}$ and $\mathcal{Z}$ be two ellipsoids in $\mathbb{R}^2$ where $\mathcal{Y} \equiv \frac{{y_1}^2}{4} + {y_2}^2 = 1$ and $\mathcal{Z} \equiv \frac{(z_1-5)^2}{4} + {z_2}^2 = 1$. If we transform to the standard form, the ellipses become

$$\mathcal{Y} = \{\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} | \left\|\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\| \le 1\}$$

and

$$\mathcal{Z} = \{\begin{pmatrix}5\\0\end{pmatrix} - \begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} | \left\|\begin{pmatrix}z_1\\z_2\end{pmatrix}\right\| \le 1\}$$

By using the dual problem, we compute the distance between $\mathcal{Y}$ and $\mathcal{Z}$ is

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = 5a_1 - 2\sqrt{4a_1^2 + a_2^2} \\ &\text{subject to } a_1^2 + a_2^2 \le 1 \end{aligned} \tag{21}$$

If we solve the problem (21) by Lagrange multiplier, we obtain that the distance between $\mathcal{Y}$ and $\mathcal{Z}$ is 1.

Next, let us consider the case when $\mathcal{Z}$ be a normed ball, that is $\mathcal{Z} = \{\boldsymbol{z_c} - \rho I\boldsymbol{v} | \|\boldsymbol{v}\| \le 1\}$. Then, the problem (17) become

$$\begin{aligned} &\text{minimize } \|\boldsymbol{y_c} - \boldsymbol{z_c} + B\boldsymbol{u} + \rho I\boldsymbol{v}\| \\ &\text{subject to} \|\boldsymbol{u}\| \le 1, \|\boldsymbol{v}\| \le 1 \end{aligned} \tag{22}$$

So, the dual of (22) has the form

$$\begin{aligned} &\text{maximize } \sigma(\boldsymbol{a}) = -\langle \boldsymbol{a}, \boldsymbol{y_c} - \boldsymbol{z_c}\rangle - \rho\|\boldsymbol{a}\|' - \|B^\dagger \boldsymbol{a}\|' \\ &\text{subject to} \|\boldsymbol{a}\|' \le 1 \end{aligned} \tag{23}$$

CONCLUSION

The minimum distance between two convex sets can be measured by determining the closest point among them. One another way is by considering Minimum Norm Duality Theorem, which is by measuring the maximum distance the supporting hyperplane that constructed to separate them. Some references have studied the minimum distance problem in Euclidean space, so they put matrix as the operator. We still apply some rule that stand on Euclidean space to Hilbert space. Further research we will consider the best approximation problem in more specific Hilbert space, such as function space (Lebesgue), matrix space, or sequence space to observe the behavior of best approximation problem in these spaces.

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