

Bounded variation functions and some properties on metric spaces

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Abstract. Bounded variation functions of a single variable were first introduced by Camille Jordan (1881). Bounded variation functions or *BV* functions is a function with the total variation is finite. The variation in a function aims to measure how much increase and decline occur in the function. In this paper, we generalize this function in the metric space. Let (X, d) be a complete metric space and $\gamma: S \rightarrow \mathbb{R}$ is a function, where S is a closed and bounded subset of X . The function γ is of bounded variation on S if $V(\gamma, S)$ is finite, i.e $V(\gamma, S) < \infty$. The result of this paper is the definition and properties algebraic of bounded variation function on metric spaces. Some properties of this function is multiplication with scalar, sum of two functions, and product of two functions. The function of bounded variation to S are also bounded variation to each subspace from S . If γ is a bounded variation function on S then γ is bounded on S . We also consider the related topic it is absolute continuity. If a function γ is absolutely continuous on S then γ is of bounded variation on S .

1. Introduction

Research about bounded variation functions is an interesting topic in the field of analysis. This study aims to sharpen and expand research topics of functions in the field of analysis. Therefore, this paper will study the bounded variation functions in the metric space and examine its properties.

Bounded variation functions or *BV* functions is a function with bounded variation. Meanwhile, the variation function is a function in which the related variables change in relation to each other. Suppose that V is a non-empty set, a function $\mu: V \rightarrow \mathbb{R}$ is said to be bounded if the function μ has an upper and lower bound, in other words, there exist a real number $K > 0$ such that $|\mu(x)| \leq K$ for each $x \in V$ [6].

Several researchers have conducted research related to variation functions, including P. Lahti in [4] conducted research on the extensions and trace of *BV* functions on metric space. Further research, namely the existence of traces of limited variable functions has been shown by P. Lahti and N. Shanmugalingam [5], in this study it was shown that the existence of trace of *BV* functions to certain class boundaries of domains in the metric space which is equipped with 1-poincare inequality, and we get L^1 trace function estimate.

The definition of *BV* functions in metric space was studied in [1]. In this study, it was explained the definition and the total variation of *BV* functions in the topology space L^1 . To study *BV* functions in metric spaces, see also [2] for the definitions of *BV* in Euclidean space \mathbb{R}^n and the definitions of *BV* in a metric measure space in [3].

Let (X, d) is a metric space, where X is a non-empty set and d is a function $X \times X$ to \mathbb{R} , such that for every $p, q, r \in X$ satisfies the four axioms in [7]. This condition can be explained as follows, X is usually



called the underlying set of (X, d) , where the member of X is the point and for each non-negative number p, q in $d(p, q)$ is the distance from p to q . In this paper, we assume that $(X, d) = X$ is a complete metric space.

Based on these studies, in this paper the author show a different approach of BV functions. The variation of functions in the metric space is closely related to the concept of supremum and infimum. This paper will also discuss the properties that apply to metric space, and the related topic it is absolutely continuous. In the last of this paper we will show the relationship of an absolute continuous function and bounded variation function.

2. Preliminaries

Before we discuss about BV functions in metric space we started a discussion about supremum, infimum, bounded variation function of a single variable, and then partitions in metric space.

2.1. Supremum and infimum

Definition 2.1.1 [6] Let M be a nonempty subset of \mathbb{R} .

- The set M is bounded above if there exist a number $x \in \mathbb{R}$ such that $m \leq x$ for all $m \in M$. Then each number x is called an upper bound of M .
- The set M is bounded below if there exist a number $y \in \mathbb{R}$ such that $y \leq m$ for all $m \in M$. Then each number x is called a lower bound of M .
- The set M is bounded if it is bounded above and bounded below. The set M said to be unbounded if it is not bounded.

After clearly defined definitions of finite sets, supremum and infimum definitions emerged.

Definition 2.1.2 [6] Let M be a nonempty subset of \mathbb{R} .

- Suppose that M is bounded above. The number α is said to be a supremum of M if α is an upper bound of M and if there is another upper bound of M . We write supremum of M with $\sup \alpha$.
- Suppose that M is bounded below. The number β is said to be a infimum of M if β is a lower bound of M and if there is another lower bound of M . We write infimum of M with $\inf \beta$.

From the definition above, it can be seen that the supremum and infimum of the set $M \subset \mathbb{R}$ are single. While the upper bound and lower bound of the set $M \subset \mathbb{R}$ are more than one.

The following statement concerning the existence of supremum. Thus, we say that \mathbb{R} is a complete ordered field.

The Completeness Property of \mathbb{R} 2.1.3 Every nonempty set of \mathbb{R} that has an upper bound also has a supremum in \mathbb{R} [6].

For the next subsection, we will explain about functions of bounded variation on real set interval.

2.2. Function of bounded variation

Before we discuss about BV functions of a single variable, we first discuss about bounded functions.

Definition 2.2.1 [6] Let A is a set. A function $g: A \rightarrow \mathbb{R}$ is said to be bounded to A if there is a constant $K > 0$ such that $|g(x)| \leq K$ for all $x \in A$.

Definition 2.2.2 [8] A partition of an interval $I = [a, b]$ is a collections $P = \{I_k\}_{k=1}^n = \{I_1, I_2, \dots, I_n\}$ from subintervals I of non overlapping such that $\bigcup_{k=1}^n I_k = [a, b]$.

From the above definition, the variation of functions can be defined.

Definition 2.2.3 [8] Let $g: [a, b] \rightarrow \mathbb{R}$ is a functions and $[c, d] \subseteq [a, b]$. The variation of the function g in $[c, d]$ is denoted by $V(g, [c, d])$ and is defined as follows

$$V(g, [c, d]) = \sup \left\{ \sum_{i=1}^n |g(x_i) - g(x_{i-1})| : x_i: 1 \leq i \leq n \text{ partition on } [c, d] \right\}$$

where the supremum is taken from all possible partition on $[c, d]$. The function g said to be bounded variation on $[c, d]$ if $V(g, [c, d])$ is finite.

Theorem 2.2.4 [8] If g is a functions of bounded variation on $[a, b]$, then g is bounded on $[a, b]$.

Proof. Given g is bounded variation on $[a, b]$, then

$$V(g, [a, b]) = \sup \sum_{i=1}^n |f(d_i) - f(c_i)| < \infty,$$

Where $c_i, d_i: 1 \leq i \leq n$ is a partition on $[c, d]$.

We will prove that g is bounded on $[a, b]$, there is $K > 0$ such that $|g(x)| \leq K$, for all $x \in [a, b]$, then

$$|g(x) - g(a)| + |g(b) - g(x)| \leq V(g, [a, b]),$$

with $|g(a)|$ and $|g(b)|$ bounded on $[a, b]$. Applies

$$\begin{aligned} |g(x)| &= |g(x) - g(a) + g(a)| \\ &\leq |g(x) - g(a)| + |g(a)| \\ &\leq |g(x) - g(a)| + |g(x) - g(b)| + |g(a)| + |g(b)| \\ &\leq V(g, [a, b]) + |g(a)| + |g(b)| \\ &= M < \infty. \end{aligned}$$

So we got that g bounded on $[a, b]$.

2.3. Metric space

We explain about the metric spaces and partitions on metric space.

Definition 2.3.1 [7] Let X be a non-empty set. A metric d from $X \times X$ to \mathbb{R} such that for all $p, q, r \in X$ we have:

1. $d(p, q) \geq 0$
2. $d(p, q) = 0$ if and only if $p = q$.
3. $d(p, q) = d(q, p)$ (Symmetry).
4. $d(p, q) \leq d(p, r) + d(r, q)$ (triangle inequality)

The pair of the set X and metric d (X, d) is called metric space.

For the next, we will explain about the partition on metric spaces.

Definition 2.3.2 [9] A collection $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^n$ of ordered pairs (points cells) is said to be a partition of Q if Q_1, Q_2, \dots, Q_n are pairwise non-overlapping elements of $S \subset (X, d)$ i.e $Q = \{Q_i | Q_i \subset S, i = 1, \dots, n\}$ such that $\cup_{i=1}^n Q_i = Q$.

3. Result and discussion

This section will explain the definitions, theorems, and the algebraic properties of bounded variation functions on metric space. Let $X = (X, d)$ be a complete metric space and S be a closed bounded subset of X .

Definition 3.1 Let $\gamma: S \rightarrow \mathbb{R}$ be a function, variation of γ on S defined to be

$$V(\gamma, S) = \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\}.$$

Supremum is taken from all possible partition on S . The function γ is said to be bounded variation on S if $V(\gamma, S)$ is finite. We say that the set of all bounded variation functions in S by $BV(S)$.

Theorem 3.2 If a function $\gamma: S \rightarrow \mathbb{R}$ is bounded variation on S then γ is bounded variation function on every subspace of S .

Proof. Given that γ is a function of bounded variation on S then

$$V(\gamma, S) = \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\}.$$

Suppose that E is a subspace of S , $E \subseteq S$, by the definition, we get $V(\gamma, E) \leq V(\gamma, S)$ for each $E \subseteq S$. So, the function γ is of bounded variation on every subspace of S .

The following describes the algebraic properties of bounded variation functions on metric space.

Theorem 3.3 Let $\gamma: S \rightarrow \mathbb{R}$ is a function of bounded variation on S then γ is of bounded on S .

Proof. We will proof this theorem by contradiction. Suppose that γ is infinite in S , then there exist $c \in S$ that satisfy

$$\lim_{x \rightarrow c} |\gamma(x)| = \infty,$$

Such that for each partition $Q = \{(Q_i, x_i) | i = 1, 2, \dots, n\}$ on S there exist Q_i containing c and

$$\left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| = \infty,$$

So that

$$\begin{aligned} V(\gamma, S) &= \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\ &= \infty. \end{aligned}$$

We get a contradiction so that the function γ is bounded to S .

Theorem 3.4 Let γ is bounded variation function on S and k be a constant. Then $k\gamma$ is bounded variation on S .

Proof. Given that $\gamma \in BV(S)$, for each partition on S , $Q = \{Q_i | Q_i \subset S, i = 1, \dots, n\}$ such that $V(\gamma, S) < \infty$. Then

$$\begin{aligned} V(k\gamma, S) &= \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} k\gamma(x) - \inf_{x \in Q_i} k\gamma(x) \right| \right\} \\ &\leq \sup_Q \left\{ \sum_{i=1}^n |k| \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\ &\leq |k| \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\ &\leq |k| V(\gamma, S) \\ &< \infty. \end{aligned}$$

For any partition, we get $k\gamma$ is bounded variation. Furthermore, we can observe that $V(k\gamma, S) = |k|V(\gamma, S)$.

Next we will explain about addition of two function of bounded variation.

Theorem 3.6. Let γ and φ be a functions of bounded variation on S , then $\gamma + \varphi$ is of bounded variation on S .

Proof. Take any partition $Q = \{Q_i | Q_i \subset S, i = 1, \dots, n\}$ on S , since $\gamma \in BV(S)$ and $\varphi \in BV(S)$ then $V(\gamma, S) < \infty$ and $V(\varphi, S) < \infty$, such that $\gamma + \varphi \in BV(S)$ then

$$\begin{aligned} V(\gamma + \varphi, S) &= \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} (\gamma + \varphi)(x) - \inf_{x \in Q_i} (\gamma + \varphi)(x) \right| \right\} \\ &\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} [\gamma(x) + \varphi(x)] - \inf_{x \in Q_i} [\gamma(x) + \varphi(x)] \right| \right\} \\ &\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) + \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \varphi(x) \right| \right\} \\ &\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) + \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right| \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \left[\sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right] + \left[\sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right] \right| \right\} \\
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\
&\quad + \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right| \right\} \\
&\leq V(\gamma, S) + V(\varphi, S) \\
&\leq \infty.
\end{aligned}$$

Because the partition was arbitrary and we get that $V(\gamma + \varphi, S) = V(\gamma, S) + V(\varphi, S)$ is finite, by axiom 2.1.3. the sum of functions $\gamma + \varphi$ is of bounded variation.

The next theorem is the product of two bounded variation functions.

Theorem 3.6 Let γ and φ be a functions of bounded variation on S , then $\gamma\varphi$ is of bounded variation on S .

Proof. Based on the theorem 3.4. Functions γ and φ are bounded on S , it means that there exist $K_\gamma > 0$ which satisfies $|\gamma(x)| \leq K_\gamma$ and there exist $K_\varphi > 0$ which satisfies $|\varphi(x)| \leq K_\varphi$ so that for each partition $Q = \{(Q_i, x_i) | i = 1, 2, \dots, n\}$ on S then

$$V(\gamma, S) < \infty \text{ and } V(\varphi, S) < \infty.$$

We will prove that $\gamma\varphi$ is a functions of bounded on S ,

$$\begin{aligned}
V(\gamma\varphi, S) &= \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma\varphi(x) - \inf_{x \in Q_i} \gamma\varphi(x) \right| \right\} \\
&= \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} [\gamma(x)\varphi(x)] - \inf_{x \in Q_i} [\gamma(x)\varphi(x)] \right| \right\} \\
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \gamma(x) \inf_{x \in Q_i} \varphi(x) \right| \right\} \\
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) \sup_{x \in Q_i} \varphi(x) \right. \right. \\
&\quad \left. \left. - \inf_{x \in Q_i} \varphi(x) \sup_{x \in Q_i} \gamma(x) + \inf_{x \in Q_i} \varphi(x) \sup_{x \in Q_i} \gamma(x) \right. \right. \\
&\quad \left. \left. - \inf_{x \in Q_i} \gamma(x) \inf_{x \in Q_i} \varphi(x) \right| \right\} \\
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) \left[\sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right] \right. \right. \\
&\quad \left. \left. + \inf_{x \in Q_i} \varphi(x) \left[\sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right] \right| \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) \left[\sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right] \right| \right\} \\
&\quad + \sup_Q \left\{ \sum_{i=1}^n \left| \inf_{x \in Q_i} \varphi(x) \left[\sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right] \right| \right\} \\
&\leq \sup_Q \left\{ \sum_{i=1}^n |K_\gamma| \left| \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right| \right\} \\
&\quad + \sup_Q \left\{ \sum_{i=1}^n |K_\varphi| \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\
&\leq |K_\gamma| \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \varphi(x) - \inf_{x \in Q_i} \varphi(x) \right| \right\} \\
&\quad + |K_\varphi| \sup_Q \left\{ \sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| \right\} \\
&\leq |K_\gamma| V(\varphi, S) + |K_\varphi| V(\gamma, S) \\
&\leq |K_\varphi| V(\gamma, S) + |K_\gamma| V(\varphi, S) \\
&< \infty.
\end{aligned}$$

Because the partition was arbitrary, by the axiom 2.1.3 the multiple of functions $\gamma\varphi$ is of bounded variation.

For the next we will discuss the relationship of bounded variation functions and absolute continuous functions. First, begins with the definition of the absolute continuous function.

Definition 3.7 Let (X, d) be a complete metric space and $S \subset X$ is a compact set. The function $\gamma: S \rightarrow \mathbb{R}$ is absolutely continuous on S if for each $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ whenever for each partition $Q = \{(Q_i, x_i) | i = 1, 2, \dots, n\}$ on S such that

$$\sum_{i=1}^n \|Q_i\| < \delta(\varepsilon).$$

then

$$\sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| < \varepsilon.$$

Theorem 3.8 If $\gamma: S \rightarrow \mathbb{R}$ is absolutely continuous on S , then γ is of bounded variation on S .

Proof. Let $Q \subseteq X$, with $\|Q\| = \sup\{d(x, y) | x, y \in Q\}$, because S is a compact set, there is a finite cover $\mathcal{G} = \{G_i | i = 1, \dots, n\}$ then $S \subseteq \cup_{i=1}^n G_i$. Take $Q_i \subseteq G_i$, with $Q_i \cap Q_j = \emptyset$, Q_i compact, and $\cup_{i=1}^n Q_i = S$.

Because it is known that γ is an absolute continuous function, then for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for each partition $Q = \{(Q_i, x_i) | i = 1, 2, \dots, n\}$ in S which satisfies $\sum_{i=1}^n \|Q_i\| < \delta(\varepsilon)$ then $\sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| < \varepsilon$. Take $\|Q\| \subset \delta(\varepsilon)$ and $k = \frac{\|S\|}{\delta(\varepsilon)}$.

therefore $\|Q\| \subset \delta(\varepsilon)$ then $\left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| < \varepsilon$. For any $i = 1, 2, \dots, k$ then

$$\sum_{i=1}^n \left| \sup_{x \in Q_i} \gamma(x) - \inf_{x \in Q_i} \gamma(x) \right| < k\varepsilon = \varepsilon'.$$

So it is proved that γ is of bounded variation on S .

4. Conclusion

In this research we examined definition and provide proofs some algebraic properties of BV functions on metric spaces. The functions of bounded variation to S are also bounded variation to each subspace from S . If γ be function of bounded variation on S then γ is bounded on S . The algebraic properties that we prove are multiplication with scalar, sum of two functions, and product of two functions. In the last result, we also discuss the related topics it is absolute continuity. We also proved that γ is absolutely continuous then γ is also bounded variation functions.

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