Applied Drazin Inverse to Moore-Penrose inverse in rings with involution

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Applied Drazin Inverse to Moore-Penrose inverse in rings with involution

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Abstract. Moore-Penrose inverse is one of development of generalized inverse. In this paper, we defined and studied a relation between the Moore-Penrose inverse and the Drazin inverse in the setting of rings with involution. The results of this paper are new characterizations of Moore-Penrose inverse by applying Drazin inverse with an algebraic proof.

1. Introduction

Moore-Penrose inverse was independently described by Moore (1920) and Penrose (1955). In numerical linear algebra, the Moore-Penrose inverse is commonly to find the matrix inverse of a singular matrix or rectangular matrix. Besides wide uses in the generalized matrix, Moore-Penrose inverse has also been applied in C*-algebras (Harte ,1992) , C*-crosses product (Hiroyuki Osaka, 1998) , and many more. Many types characteristic of Moore-Penrose Inverse has been found (Dragan S. Djordjevic, Pedro Particioa, etc.). At this paper, we aimed to develop characteristics Moore-Penrose inverse in symmetric elements by Drazin inverse.

The paper is motivated by Mosaic and Djordjevic [2]. It was reported Moore-Penrose inverse had several characteristics including $a^{\dagger}=\left(a^{*}a\right)^{\dagger}a^{*}=a^{*}\left(aa^{*}\right)^{\dagger}$ and $\left(a^{*}\right)^{\dagger}=a\left(a^{*}a\right)^{\dagger}=\left(aa^{*}\right)^{\dagger}a$ and $\left(aa^{*}a\right)^{\dagger}=a^{\dagger}\left(a^{*}\right)^{\dagger}a^{\dagger}$ that changed in the form of Drazin. Furthermore, the form of Drazin is applied to Moore-Penrose inverse with the symmetric rule.

Let R be a ring with identity and $a\in R$. An Involution "*" in a ring is a unary operation $a\to a^{*}$ such that :

$$\left(a^{*}\right)^{*}=a \qquad (ab)^{*}=b^{*}a^{*} \qquad (a+b)^{*}=a^{*}+b^{*}$$

for all elements $a,b\in R$. An element $a\in R$ satisfying $a=a^{*}$ is called Hermitian (or symmetric).

Let $a\in R$ is said to be Moore-Penrose (MP) invertible with respect to "$\dagger$" , if there exist $a^{\dagger}\in R$ such that :

$$aa^{\dagger}a=a \qquad a^{\dagger}aa^{\dagger}=a^{\dagger} \qquad \left(aa^{\dagger}\right)^{*}=aa^{\dagger} \qquad \left(a^{\dagger}a\right)^{*}=a^{\dagger}a$$

Also, the group inverse of $a\in R$ exists if there is a $a^{\#}\in R$ such that :

$$aa^{\#}a=a \qquad a^{\#}aa^{\#}=a \qquad aa^{\#}=a^{\#}a$$

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If the group inverse exists then it is unique.

An element $a \in R$ is said to have a Drazin Inverse if there exists $a^D \in R$ such that

$$a^D a a^D = a^D \qquad a a^D = a^D a \qquad a^{k+1} a^D = a^k \text{ ,for some non-negative integer k}$$

If $a \in R$ has a Drazin Inverse, then the smallest possible non-negative integer involve in (3) is called the Drazin index of a.

As for group inverse and Moore-Penrose inverse, if the Drazin inverse exists then it is unique.

2. Applied Drazin Inverse

In this section, we construct several lemmas that used to apply Drazin inverse to Moore Penrose inverse. In the paper of Koliha and Patricio [1] in theorem 5.4 said that Moore-Penrose invertible symmetric element is Drazin inverse. Based on that theorem, we applied Drazin inverse to Moore Penrose inverse that produces new characteristics. Before we apply Drazin inverse to Moore-Penrose inverse, we constructed some properties about Drazin inverse. The following lemma constructed by applied symmetric theorem in Drazin inverse. [4]

Lemma 2.1 Let R be a ring with involution. If $a \in R^\dagger$ then $(a^D)^* = (a^*)^D$

Proof Let $a \in R^\dagger$, first

$$\left(a^D\right)^* a^* \left(a^D\right)^* = \left(a^D\right)^* \left(a^D a\right)^* = \left(a^D a a^D\right)^* = \left(a^D\right)^*,$$

second

$$a^* \left(a^D\right)^* = \left(a^D a\right)^* = \left(a a^D\right)^* = \left(a^D\right)^* a^*,$$

third

$$\left(a^{k+1}\right)^* \left(a^D\right)^* = \left(a^D a^{k+1}\right)^* = \left(a^{k+1} a^D\right)^* = \left(a^k\right)^*$$

We conclude that $(a^D)^* = (a^*)^D$ ■

The following theorem lemma presents a relation between Moore-Penrose inverse and Drazin inverse.

Theorem 2.2 [1] Let R be a ring with involution and $a \in R^\dagger$. If $a^* = a$, then $a^\dagger = a^D$.

Proof. Suppose $a \in R^\dagger$ and $a^* = a$, first

$$a a^D a = a a a^D = a^2 a^D = a$$

second

$$a^D a a^D = a^D$$

third

$$\left(a a^D\right)^* = \left(a^D\right)^* a^* = a^D a = a a^D$$

fourth

$$\left(a^D a\right)^* = a^* \left(a^D\right)^* = a a^D = a^D a$$

Because all of the Moore-Penrose rules has been satisfied, so we can conclude that $a^\dagger = a^D$, if $a^* = a$. ■

Futhermore, we are modifying theorem 2.2 became to lemma 2.3.

Lemma 2.3 Let R be a ring with involution. If $a \in R^\dagger$, then $\left(a^* a\right)^\dagger = \left(a^* a\right)^D$.

Proof. Let $a \in R^\dagger$, we get

$$a^* a \left(a^* a\right)^D a^* a = a^* a a^* \left(a^* a\right)^D a = a^* a a^* a \left(a^* a\right)^D = \left(a^* a\right)^2 \left(a^* a\right)^D = a^* a,$$

and

$$\left(a^* a\right)^D a^* a \left(a^* a\right)^D = \left(a^* a\right)^D,$$

also

$$\left(a^*a\left(a^*a\right)^D\right)^* = \left(\left(a^*a\right)^D\right)^*\left(a^*a\right)^* = \left(\left(a^*a\right)^*\right)^D\left(a^*a\right)^* = \left(a^*\left(a^*\right)^*\right)^D a^*\left(a^*\right)^* = \left(a^*a\right)^D a^*a$$

$$= a^*\left(a^*a\right)^D a = a^*a\left(a^*a\right)^D$$

furthermore

$$\left(\left(a^*a\right)^D a^*a\right)^* = \left(a^*a\right)^*\left(\left(a^*a\right)^D\right)^* = \left(a^*a\right)^*\left(\left(a^*a\right)^*\right)^D = a^*\left(a^*\right)^*\left(a^*\left(a^*\right)^*\right)^D = a^*a\left(a^*a\right)^D$$

$$= a^*\left(a^*a\right)^D a = \left(a^*a\right)^D a^*a$$

First rule, second rule, third rule, and fourth rule of Moore-Penrose inverse has been satisfied, we can conclude that $\left(a^*a\right)^\dagger = \left(a^*a\right)^D$. ■

Mosaic and Djordjevic [2] verified that $a^\dagger = \left(a^*a\right)^\dagger a^* = a^*\left(aa^*\right)^\dagger$ in Theorem 1.2 (g). By using lemma 2.3 and Theorem 1.2 (g) in [2] we get Theorem 2.4.

Theorem 2.4

Let R be a ring with involution. If $a \in R^\dagger$, then $a^\dagger = \left(a^*a\right)^D a^* = a^*\left(aa^*\right)^D$

Proof. Let $a \in R^\dagger$, we get

$$a^\dagger = a^\dagger aa^\dagger = a^\dagger\left(aa^\dagger\right)^* = a^\dagger\left(a^\dagger\right)^* a^* = \left(a^*a\right)^\dagger a^* = \left(a^*a\right)^D a^*$$

and

$$a^\dagger = a^\dagger aa^\dagger = \left(a^\dagger a\right)^* a^\dagger = a^*\left(a^\dagger\right)^* a^\dagger = a^*\left(aa^*\right)^\dagger = a^*\left(aa^*\right)^D$$

We conclude that $a^\dagger = \left(a^*a\right)^D a^* = a^*\left(aa^*\right)^D$ ■

Mosaic and Dordjevic said [2] reported that $\left(a^*\right)^\dagger = a\left(a^*a\right)^\dagger = \left(aa^*\right)^\dagger a$ in Theorem 1.2 (h). By using Theorem 2.4 and Theorem 1.2 (h) in [2] on we get Theorem 2.5 .

Theorem 2.5

Let R be a ring with involution. If $a \in R^\dagger$, then $\left(a^*\right)^\dagger = a\left(a^*a\right)^D = \left(aa^*\right)^D a$

Proof. Let $a \in R^\dagger$, we get

$$\left(a^*\right)^\dagger = a\left(a^*a\right)^\dagger = aa^\dagger\left(a^\dagger\right)^* = aa^\dagger\left(a^*\right)^\dagger = a\left(a^*a\right)^D a^*\left(a^*\left(a^*\right)^*\right)^D\left(a^*\right)^* = a\left(a^*a\right)^D a^*\left(a^*a\right)^D a$$

$$= a\left(a^*a\right)^D a^*a\left(a^*a\right)^D = a\left(a^*a\right)^D$$

and

$$\left(a^*\right)^\dagger = \left(aa^*\right)^\dagger a = \left(a^\dagger\right)^* a^\dagger a = \left(a^*\right)^\dagger a^\dagger a = \left(\left(a^*\right)^* a^*\right)^D\left(a^*\right)^*\left(aa^*\right)^D a^*a = \left(aa^*\right)^D a\left(aa^*\right)^D a^*a$$

$$= \left(aa^*\right)^D aa^*\left(aa^*\right)^D a = \left(aa^*\right)^D a$$

We conclude that $\left(a^*\right)^\dagger = a\left(a^*a\right)^D = \left(aa^*\right)^D a$ ■

Mosaic and Dordjevic said in lemma 2.1 that $\left(aa^*a\right)^\dagger = a^\dagger\left(a^*\right)^\dagger a^\dagger$. By using Theorem 2.2 and lemma 2.1 in [2] we get Theorem 2.6 .

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Theorem 2.6

Let R be a ring with involution. If $a \in R^\dagger$, then $\left(aa^*a\right)^\dagger = a^D\left(a^*\right)^D a^D$

Proof. Let $a \in R^\dagger$, we get

First

$$aa^*a\left(aa^*a\right)^D aa^*a = aa^*aa^D\left(a^*\right)^D a^Daa^*a = aa^*a\left(a^*\right)^D a^Da^*a = a^2\left(a^*\right)^2\left(a^D\right)^* a^Da$$
$$= a^2\left(a^Da^2\right)^* a^Da = a^2\left(a\right)^* a^Da = a^2a^Da^*a = aa^*a$$

Second

$$\left(aa^*a\right)^D aa^*a\left(aa^*a\right)^D = a^D\left(a^*\right)^D a^Daa^*aa^D\left(a^*\right)^D a^D = \left(a^*\right)^D a^Daa^Da^*a^Daa^D\left(a^*\right)^D$$
$$= \left(a^*\right)^D a^Da^*a^D\left(a^*\right)^D = a^D\left(a^Daa^D\right)^* a^D = a^D\left(a^D\right)^* a^D = a^D\left(a^*\right)^D a^D$$

Third

$$\left(aa^*a\left(aa^*a\right)^D\right)^* = \left(a^D\left(a^*\right)^D a^D\right)^*\left(aa^*a\right)^* = a^D\left(a^D\right)^* a^*\left(a^D\right)^* aa^* = a^D\left(a^Daa^D\right)^* aa^* = a^D\left(a^D\right)^* aa^*$$
$$= aa^*a^D\left(a^*\right)^D = aa^*a^Daa^D\left(a^*\right)^D = aa^*aa^D\left(aa^*\right)^D = aa^*a\left(aa^*a\right)^D$$

Fourth

$$\left(\left(aa^*a\right)^D aa^*a\right)^* = \left(aa^*a\right)^*\left(a^D\left(a^*\right)^D a^D\right)^* = \left(a^*a\right)^* a^*\left(a^D\right)^*\left(a^D\left(a^*\right)^D\right)^* = a^*a\left(a^D\right)^* a^*\left(a^D\right)^* a^D$$
$$= a^*a\left(a^Daa^D\right)^* a^D = a^*a\left(a^D\right)^* a^D = \left(a^*\right)^D a^Da^*a = a^D\left(a^*\right)^D a^Daa^*a = \left(aa^*a\right)^D aa^*a$$

So Moore-Penrose inverse has been satisfied, we can conclude that $\left(aa^*a\right)^\dagger = a^D\left(a^*\right)^D a^D$ ■

3. Conclusion

Moore-Penrose inverse is very important in finding matrix inverse that singular matrix or rectangular matrix. There are many characters in Moore-Penrose inverse, but there is no developments in recent years. In this paper, we constructed new characteristic in Moore-Penrose by applied Drazin inverse in Moore-Penrose inverse with algebraic proof. The results are $a^\dagger = \left(a^*a\right)^D a^* = a^*\left(aa^*\right)^D$ and $\left(a^*\right)^\dagger = a\left(a^*a\right)^D = \left(aa^*\right)^D a$ also $\left(aa^*a\right)^\dagger = a^D\left(a^*\right)^D a^D$.

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