Linear Relation on Hilbert Space

by Susilo Hariyanto

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Abstract. Given $\mathfrak{H}$ be Hilbert space over $\mathbb{C}$. If $\mathfrak{H}$ is a Hilbert space then $\mathfrak{H}^2$ is also Hilbert space. A linear relation on $\mathfrak{H}$ is a subspace of $\mathfrak{H}^2$. A linear relation can be multivalued part or not multivalued part. This paper proposes to discuss and show terms that a linear relation is a bounded linear operator and its spectrum analysis. We give the result that if $\mathfrak{R}$ is an injective relation on $\mathfrak{H}$ and range of $\mathfrak{R}$ is dense then $\mathfrak{R}^*$ is also injective and $(\mathfrak{R}^*)^{-1} = (\mathfrak{R}^{-1})^*$. Consequently, if a relation $\mathfrak{R}$ on $\mathfrak{H}$ is an injective and self-adjoint, then a relation $\mathfrak{R}^{-1}$ is also self-adjoint. If a relation $\mathfrak{R}$ is a symmetric on $\mathfrak{H}$ then $N(z-\mathfrak{R}^*) = R(z-\mathfrak{R})^{\perp} \cap D(z-\mathfrak{R}^*)$ and $N(z-\mathfrak{R}) = R(z-\mathfrak{R}^*)^{\perp} \cap D(z-\mathfrak{R})$. If a relation $\mathfrak{R}$ is a symmetric and $z \in \mathbb{C}$, then $\|z\mathfrak{a} - \mathfrak{b}\|^2 \geq q^2 \|\mathfrak{a}\|^2$. If a relation $\mathfrak{R}$ is closed, bounded and $|z| \geq \|\mathfrak{b}\|$, then $z \in \rho(\mathfrak{R})$. Consequently, if a relation $\mathfrak{R}$ is closed, bounded and $|z| \leq \|\mathfrak{b}\|$, then $z \in \sigma(\mathfrak{R})$.

INTRODUCTION

A linear relation was first introduced by Arens in [1]. Since then the interest in studying linear relation has evolved, supposing Kascic [2], Acharya [3], Langer and Textorious [4], Sandovici and Sebestyen [5], Gheorghe and Vasilescu [6], Hassi et al [7], Miranda and Labrousse[8], Baskakov and Chernyschov [9], Popovici and Sebestyen [10], and Dijksma and de Snoo [11].

Let be $\mathfrak{H}$ Hilbert space over field $\mathbb{C}$ and $\mathfrak{H}^2$ is a Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$. A linear relation $\mathfrak{R}$ on $\mathfrak{H}$ is a set of pairs $\mathfrak{a}$ and $\mathfrak{b}$ elements with $\mathfrak{a}, \mathfrak{b} \in \mathfrak{H}$ and denoted $\mathfrak{R} = \{(\mathfrak{a}, \mathfrak{b}) : \mathfrak{a}, \mathfrak{b} \in \mathfrak{H}\}$. A linear relation $\mathfrak{R}$ is a subspace of $\mathfrak{H}^2$. One example of a linear relation is a graph of an operator. A linear relation can be a multivalued part or not multivalued part.

A function on $\mathfrak{H}$ is a relation on $\mathfrak{H}$ in which for each $\mathfrak{a} \in \mathfrak{H}$ there exists a unique $\mathfrak{b} \in \mathfrak{H}$. An operator is a function from one vector space to another vector space. An operator T is called linear if for every $\mathfrak{a}, \mathfrak{b} \in \mathfrak{H}$ and any scalar $\beta \in \mathbb{C}$ apply $T(\mathfrak{a}+\mathfrak{b}) = T(\mathfrak{a}) + T(\mathfrak{b})$ and $T(\beta\mathfrak{a}) = \beta T(\mathfrak{a})$. Furthermore, an operator T is called bounded if there exists $C \geq 0$ so that $\|T(\mathfrak{a})\| \leq C\|\mathfrak{a}\|$.

A relation has been used eigenvalue problem in a differential equation. As an example [3], consider the canonical systems

$$Mp'(x) = zK(x)p(x) \tag{1}$$

where $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $K(x)$ is a 2x2 positive semidefinite matrix whose entries are locally integrable, induced a multivalued linear relation. The above canonical system can be written in the form

$$K(x)^{-1} Mp' = zp \tag{2}$$

and $K(x)$ can be regarded as an operator on Hilbert space. If $K(x)$ is not invertible then cannot be regarded as an operator.

The paper is divided into several sections as follows. In section 2 we present notations of a linear relation used and give some characteristics of linear relation on Hilbert space. In section 3 we generalize that a linear relation is a bounded linear operator and show its spectral analysis.

LINEAR RELATION

The following notations of relation $\mathfrak{R}$ on $\mathfrak{H}$ can be found [1, 2, 5, 6]. A linear relation $\mathfrak{R}$ on $\mathfrak{H}$ is a set of pairs $\mathfrak{a}$ and $\mathfrak{b}$ elements with $\mathfrak{a},\mathfrak{b} \in \mathfrak{H}$ and denoted $\mathfrak{R} = \{(\mathfrak{a},\mathfrak{b}) : \mathfrak{a},\mathfrak{b} \in \mathfrak{H}\}$ .We will omit the word "linear" because we consider all relations in this paper are linear. A relation on $\mathfrak{H}$ is a subspace of $\mathfrak{H}^2$. The definition of the domain and range of $\mathfrak{R}$ respectively, are

$$D(\mathfrak{R}) = \{\mathfrak{a} \in \mathfrak{H} : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\},\ R(\mathfrak{R}) = \{\mathfrak{b} \in \mathfrak{H} : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}. \tag{3}$$

The definition of the kernel and multivalued part of $\mathfrak{R}$ respectively, are

$$N(\mathfrak{R}) = \{\mathfrak{a} \in \mathfrak{H} : (\mathfrak{a},0) \in \mathfrak{R}\},\ M(\mathfrak{R}) = \{\mathfrak{b} \in \mathfrak{H} : (0,\mathfrak{b}) \in \mathfrak{R}\}. \tag{4}$$

The inverse relation is a relation on $\mathfrak{H}$ that is obtained from by exchanging a component of the elements of $\mathfrak{R}$. The inverse relation is denoted by $\mathfrak{R}^{-1} = \{(\mathfrak{b},\mathfrak{a}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}$. The duality of $\mathfrak{R}$ and its inverse $\mathfrak{R}^{-1}$ is denoted by

$$D(\mathfrak{R}^{-1}) = R(\mathfrak{R}),\ R(\mathfrak{R}^{-1}) = D(\mathfrak{R}),\ N(\mathfrak{R}^{-1}) = M(\mathfrak{R}),\ M(\mathfrak{R}^{-1}) = N(\mathfrak{R}). \tag{5}$$

Adjoint of relation $\mathfrak{R}$ on $\mathfrak{H}$ is a closed relation defined by

$$\mathfrak{R}^* = \{(\mathfrak{g},\mathfrak{h}) \in \mathfrak{H}^2 : \langle \mathfrak{b},\mathfrak{g} \rangle = \langle \mathfrak{a},\mathfrak{h} \rangle, \forall (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}. \tag{6}$$

Sum of relations $\mathfrak{R}, \mathcal{S}$ on $\mathfrak{H}$ is defined by

$$\mathfrak{R} + \mathcal{S} = \{(\mathfrak{a},\mathfrak{b}+\mathfrak{h}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}, (\mathfrak{a},\mathfrak{h}) \in \mathcal{S}\}. \tag{7}$$

Product of relations $\mathfrak{R}, \mathcal{S}$ on $\mathfrak{H}$ is defined by

$$\mathcal{S}\mathfrak{R} = \{(\mathfrak{a},\mathfrak{h}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}, (\mathfrak{a},\mathfrak{h}) \in \mathcal{S}\}. \tag{8}$$

If $\mathfrak{R}$ is linear then $\mathfrak{R}^*$ is also linear. If $\mathfrak{R}, \mathcal{S}$ is linear then $\mathfrak{R} + \mathcal{S}$ and $\mathcal{S}\mathfrak{R}$ are also linear. For any z is an element of the field, we put

$$z - \mathfrak{R} = \{(\mathfrak{a}, z\mathfrak{a} - \mathfrak{b}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}, z \in \mathbb{C}\}, \tag{9}$$

$$z\mathfrak{R} = \{(\mathfrak{a}, z\mathfrak{b}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}, z \in \mathbb{C}\}. \tag{10}$$

The basic proporties of relation can be found in [1]. We give the following properties about the adjoint.

$$\mathfrak{R} \subset \mathcal{S} \Rightarrow \mathcal{S}^* \subset \mathfrak{R}^*, \tag{11}$$

$$(z\mathfrak{R})^* = \bar{z}\mathfrak{R}^*, \tag{12}$$

$$(\mathfrak{R}^{-1})^* = (\mathfrak{R}^*)^{-1}, \tag{13}$$

$$\mathfrak{R}^* \text{ is closed, } \mathfrak{R} = \mathfrak{R}^{**}, \tag{14}$$

$$M(\mathfrak{R}^*) = R(\mathfrak{R}^*)^{\perp}. \tag{15}$$

A relation $\mathfrak{R}$ is called symmetric if $\mathfrak{R} \subset \mathfrak{R}^*$ and self-adjoint if $\mathfrak{R} = \mathfrak{R}^*$. Every self-adjoint relation is symmetric. A relation is a isometry if $\langle \mathfrak{a},\mathfrak{c} \rangle = \langle \mathfrak{b}, d \rangle, \forall (\mathfrak{a},\mathfrak{b}), (\mathfrak{c}, d) \in \mathfrak{R}$. If relation $\mathfrak{R}$ is an isometry and $D(\mathfrak{R}) = R(\mathfrak{R}) = \mathfrak{H}$ then $\mathfrak{R}$ is unitary. A relation is an injective if $N(\mathfrak{R}) = \{0\}$ and a relation is a surjective if $R(\mathfrak{R}) = \mathfrak{H}$. If a relation $\mathfrak{R}$ is an injective and surjective then a relation $\mathfrak{R}$ is called bijective. A relation $\mathfrak{R}$ is a graph of an operator if and only if relation $\mathfrak{R}$ apply $M(\mathfrak{R}) = \{0\}$.

Theorem 1 Let $\mathfrak{R}$ is an injective relation on $\mathfrak{H}$. If range of $\mathfrak{R}$ is dense then $\mathfrak{R}^*$ is also injective and $(\mathfrak{R}^*)^{-1} = (\mathfrak{R}^{-1})^*$.

Proof. $R(\mathfrak{R})$ is dense, that is $R(\mathfrak{R})^{\perp} = N(\mathfrak{R}^*) = N(\mathfrak{R}) = \{0\}$. Consequently, $\mathfrak{R}^*$ is an injective relation and $(\mathfrak{R}^*)^{-1} = (\mathfrak{R}^{-1})^*$.

Corollary 2 Let relation $\mathfrak{R}$ on $\mathfrak{H}$ is an injective and self-adjoint. Then relation $\mathfrak{R}^{-1}$ is also self-adjoint.

Theorem 3 Let relation $\mathfrak{R}$ is a symmetric on $\mathfrak{H}$. Then

1. $N(z-\mathfrak{R}^*) = R(z-\mathfrak{R})^{\perp} \cap D(z-\mathfrak{R}^*)$
2. $N(z-\mathfrak{R}) = R(z-\mathfrak{R}^*)^{\perp} \cap D(z-\mathfrak{R})$

Proof. (1) Let $(z-\mathfrak{R}^*) = \{(\mathfrak{g}, z\mathfrak{g}-\mathfrak{h}) : (\mathfrak{g},\mathfrak{h}) \in \mathfrak{R}^*\}$, we get $D(z-\mathfrak{R}^*) = \{\mathfrak{g}\}$ and $N(z-\mathfrak{R}^*) = \{\mathfrak{g} : (\mathfrak{g}, z\mathfrak{g}) \in (z-\mathfrak{R}^*)\}$. Since $R(z-\mathfrak{R})^{\perp} = \{\mathfrak{g} : \langle z\mathfrak{a}-\mathfrak{b}, \mathfrak{g}\rangle = 0, \forall(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}$ we get

$$N(z-\mathfrak{R}^*) = R(z-\mathfrak{R})^{\perp} \cap D(z-\mathfrak{R}^*). \tag{16}$$

(2) Let $(z-\mathfrak{R}) = \{(\mathfrak{a}, z\mathfrak{a}-\mathfrak{b}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}^*\}$, we get $D(z-\mathfrak{R}) = \{\mathfrak{a}\}$ and $N(z-\mathfrak{R}) = \{\mathfrak{a} : (\mathfrak{a}, z\mathfrak{a}) \in (z\text{-}\mathfrak{R})\}$. Furthermore, we have $R(z-\mathfrak{R}^*)^{\perp} = \{\mathfrak{a} : \langle z\mathfrak{g}-\mathfrak{h}, \mathfrak{a}\rangle = 0, \forall(\mathfrak{g},\mathfrak{h}) \in \mathfrak{R}^*\}$. Thus,

$$N(z-\mathfrak{R}) = R(z-\mathfrak{R}^*)^{\perp} \cap D(z-\mathfrak{R}). \tag{17}$$

ANALYSIS AND DISCUSSION

We define the following regularity domain.

Definition 4[3] Let relation $\mathfrak{R}$ on $\mathfrak{H}$. A Regularity domain of relation $\mathfrak{R}$ is the set

$$\Gamma(\mathfrak{R}) = \{z \in \mathbb{C} : \text{there exists } C(z) > 0 \text{ so that } \|(z\mathfrak{a}-\mathfrak{b})\| \geq C(z)\|\mathfrak{a}\|, \forall(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}. \tag{18}$$

The set $S(\mathfrak{R}) = \mathbb{C} \setminus \Gamma(\mathfrak{R})$ is called the spectral kernel of $\mathfrak{R}$.

Proposition 5 [3] $z \in \Gamma(\mathfrak{R})$ if and only if $(z-\mathfrak{R})^{-1}$ is a bounded linear operator on $\mathfrak{H}$.

Proof. Clearly, $(z-\mathfrak{R})^{-1} = \{(z\mathfrak{a}-\mathfrak{b}, \mathfrak{a}) : (\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\}$. For each $z\mathfrak{a}-\mathfrak{b} \in \mathfrak{H}$ there exists a unique $\mathfrak{a} \in \mathfrak{H}$ with $(z\mathfrak{a}-\mathfrak{b}, \mathfrak{a}) \in (z-\mathfrak{R})^{-1}$, so that $(z-\mathfrak{R})^{-1}$ is a function. Let $(z\mathfrak{a}_1-\mathfrak{b}_1, \mathfrak{a}_1), (z\mathfrak{a}_2-\mathfrak{b}_2, \mathfrak{a}_2) \in (z\text{-}\mathfrak{R})^{-1}$ and scalar β then $T(z\mathfrak{a}_1-\mathfrak{b}_1+z\mathfrak{a}_2-\mathfrak{b}_2) = \mathfrak{a}_1+\mathfrak{a}_2 = T(z\mathfrak{a}_1-\mathfrak{b}_1)+T(z\mathfrak{a}_2-\mathfrak{b}_2)$ and $T(\beta(z\mathfrak{a}_1-\mathfrak{b}_1)) = \beta\mathfrak{a}_1 = \beta T(z\mathfrak{a}_1-\mathfrak{b}_1)$. Consequently, $(z-\mathfrak{R})^{-1}$ is a linear operator. Given $z \in \Gamma(\mathfrak{R})$ there exists a $C(z) > 0$ so that $\|\mathfrak{a}\| \leq \frac{1}{C(z)}\|(z\mathfrak{a}\text{-}\mathfrak{b})\|$ for each $(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$ then $(z-\mathfrak{R})^{-1}$ is a bounded operator. This proves that $(z-\mathfrak{R})^{-1}$ is a bounded linear operator on $\mathfrak{H}$. Otherwise, Since there exists $C(z) > 0$ so that $\|\mathfrak{a}\| \leq \frac{1}{C(z)}\|(z\mathfrak{a}-\mathfrak{b})\|$ for each $(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$ then $z \in \Gamma(\mathfrak{R})$.

Corollary 6 If $(z-\mathfrak{R})^{-1}$ is a bounded linear operator on $\mathfrak{H}$ then $(z-\mathfrak{R})$ is also bounded linear operator.

Proof. If $(z-\mathfrak{R})^{-1}$ is a linear operator on $\mathfrak{H}$ then $(z-\mathfrak{R})$ is also a linear operator. For each $(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$, we get $\|\mathfrak{a}\| \leq \frac{1}{C(z)}\|(z\mathfrak{a}-\mathfrak{b})\| \Leftrightarrow \|(z\mathfrak{a}-\mathfrak{b})\| \geq C(z)\|\mathfrak{a}\|$ then $(z-\mathfrak{R})$ is also a bounded operator. So, $(z-\mathfrak{R})$ is also a bounded linear operator.

This means that a $(z-\mathfrak{R})$ is called a bounded linear operator on $\mathfrak{H}$ if for some $z \in \mathbb{C}$ there exists a $C(z) > 0$ so that $\|(z\mathfrak{a}-\mathfrak{b})\| \geq C(z)\|\mathfrak{a}\|, \forall(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$.

Proposition 7 [3] If relation $\mathfrak{R}$ is a symmetric then $\mathbb{C} \setminus \mathbb{R} \subset \Gamma(\mathfrak{R})$.

Proof. Given relation $\mathfrak{R}$ is a symmetric that is $\mathfrak{R} \subset \mathfrak{R}^*$. Let $z \in \Gamma(\mathfrak{R})$ then there is a constant $C(z) > 0$ so that $\|(z\mathfrak{a}-\mathfrak{b})\| \geq C(z)\|\mathfrak{a}\|, \forall(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$. If $z \in \mathbb{C}$ then $\mathbb{C} \setminus \mathbb{R} \subset \Gamma(\mathfrak{R})$.

Proposition 8 [3] A regularity domain $\Gamma(\mathfrak{R})$ is open.

Proof. Let $z_0 \in \Gamma(\mathfrak{R})$. If $z \in \mathbb{C}$ so that $|z-z_0| < C(z_0)$, then

$$\|(z\mathfrak{a}-\mathfrak{b})\| \geq \|(z_0\mathfrak{a}-\mathfrak{b})\| - |z-z_0|\|\mathfrak{a}\| \geq (C(z_0) - |z-z_0|)\|\mathfrak{a}\| \tag{19}$$

for each $(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}$. Therefore, $z \in \Gamma(\mathfrak{R})$ then $\Gamma(\mathfrak{R})$ is open.

Let $\mathfrak{R}$ is a relation on $\mathfrak{H}$. For each $z \in \mathbb{C}$ is called an eigenvalue of $\mathfrak{R}$ if there exists $\mathfrak{a} \in D(\mathfrak{R}), \mathfrak{a} \neq 0$ so that $(\mathfrak{a}, z\mathfrak{a}) \in \mathfrak{R}$ i.e. if a relation $(z-\mathfrak{R})$ is not injective, $N(z-\mathfrak{R}) \neq \{0\}$. The element $\mathfrak{a}$ is called an eigenvector of $\mathfrak{R}$ including to an eigenvalue z. For some $z \in \mathbb{C}$ the relation $(z-\mathfrak{R})$ is an injective and $(z-\mathfrak{R})^{-1}$ is a bounded linear operator, then z includes to the resolvent set $\rho(\mathfrak{R})$ of $\mathfrak{R}$. A resolvent set is denoted

$$\rho(\mathfrak{R}) = \{z \in \mathbb{C} : (z-\mathfrak{R}) \text{ is an injective, } (z-\mathfrak{R})^{-1} \in \mathcal{B}(\mathfrak{H})\}. \tag{20}$$

The spectrum of $\mathfrak{R}$ is defined by $\sigma(\mathfrak{R}) = \mathbb{C} \setminus \rho(\mathfrak{R})$. The set $\sigma_p(\mathfrak{R})$ of all eigenvalues that is contained in $\sigma(\mathfrak{R})$ is called a point spectrum of $\mathfrak{R}$.

Example:

Given $\mathfrak{R} = \{(\mathfrak{a}, 2\mathfrak{a}), \mathfrak{a} \in \mathbb{C}\}$. Clearly, $(z - \mathfrak{R}) = \{(\mathfrak{a}, z\mathfrak{a} - 2\mathfrak{a}), \mathfrak{a} \in \mathbb{C}\}$. Let $z \in \mathbb{C}$, we get

$$\begin{aligned}\|(z\mathfrak{a}-2\mathfrak{a})\| &\geq \|z\mathfrak{a}\| - \|2\mathfrak{a}\| \\ &\geq |z|\,\|\mathfrak{a}\| - 2\|\mathfrak{a}\| \\ &\geq (|z|-2)\,\|\mathfrak{a}\|\end{aligned} \tag{21}$$

We can choose $C(z) = |z| - 2$ so that $|z| - 2 > 0 \Leftrightarrow |z| > 2$. Consequently, a regularity domain of $\mathfrak{R}$ is the set

$$\Gamma(\mathfrak{R}) = \{z \in \mathbb{C} : \text{there exist } |z| > 2 \text{ so that } \|(z\mathfrak{a}-\mathfrak{b})\| \geq (|z|-2)\,\|\mathfrak{a}\|, \forall(\mathfrak{a},\mathfrak{b}) \in \mathfrak{R}\} \tag{22}$$

and a spectral kernel of $\mathfrak{R}$ is $\mathcal{S}(\mathfrak{R}) = \mathbb{C} \setminus \Gamma(\mathfrak{R})$. A resolvent set is given $\rho(\mathfrak{R}) = \{z \in \mathbb{C} : |z| > 2\}$. The spectrum of $\mathfrak{R}$ is given $\sigma(\mathfrak{R}) = \mathbb{C} \setminus \rho(\mathfrak{R}) = \{z \in \mathbb{C} : |z| > 2\}$.

There are two conditions about the relationship among regularity domain and spectral theory:

1. If $(z-\mathfrak{R})$ is an injective and $(z-\mathfrak{R})^{-1}$ is a bounded linear operator then z is a point of regular. All points of regularity form the resolvent set $\rho(\mathfrak{R})$
2. If z is not a point of regularity then z is a point of the spectrum. All points of spectrum create the spectrum $\sigma(\mathfrak{R})$.

The following theorem,together with its proof, can be found in [3].

Theorem 9 If a relation $\mathfrak{R}$ is a self-adjoint on $\mathfrak{H}$, then $\mathcal{S}(\mathfrak{R}) = \sigma(\mathfrak{R})$.

We give the following some Theorems of spectral theory.

Theorem 10 If a relation $\mathfrak{R}$ is a symmetric and $z \in \mathbb{C}$, then

$$\|(z\mathfrak{a}-\mathfrak{b})\|^2 \geq q^2\|\mathfrak{a}\|^2 \tag{23}$$

Proof. Let $z = \mathrm{p} + i\mathrm{q}$ where $\mathrm{p}, \mathrm{q} \in \mathbb{R}$, we get

$$\begin{aligned}\|(z\mathfrak{a}-\mathfrak{b})\|^2 &= \langle z\mathfrak{a}-\mathfrak{b}, z\mathfrak{a}-\mathfrak{b}\rangle \\ &= \langle z\mathfrak{a}, z\mathfrak{a}\rangle + \langle \mathfrak{b},\mathfrak{b}\rangle - z\langle \mathfrak{a},\mathfrak{b}\rangle - \bar{z}\langle \mathfrak{a},\mathfrak{b}\rangle \\ &= (\mathrm{p}^2+\mathrm{q}^2)\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2\mathrm{p}\langle \mathfrak{a},\mathfrak{b}\rangle \\ &\geq (\mathrm{p}^2+\mathrm{q}^2)\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2|\mathrm{p}|\,\|\mathfrak{a}\|\,\|\mathfrak{b}\| \\ &\geq \mathrm{q}^2\|\mathfrak{a}\|^2\end{aligned} \tag{24}$$

Theorem 11 If a relation $\mathfrak{R}$ is closed and bounded and $|z| \geq \|\mathfrak{b}\|$, then $z \in \rho(\mathfrak{R})$.

Proof. Given $z \in \mathbb{C}$ and $|z| \geq \|\mathfrak{b}\|$. We get

$$|z| \geq \|\mathfrak{b}\| \Leftrightarrow |z|\,\|\mathfrak{a}\|^2 \geq \|\mathfrak{b}\|\,\|\mathfrak{a}\|^2$$
$$\Leftrightarrow |z|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 \geq \|\mathfrak{b}\|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2$$
$$\Leftrightarrow |z|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2p\langle \mathfrak{a},\mathfrak{b}\rangle \geq \|\mathfrak{b}\|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2p\langle \mathfrak{a},\mathfrak{b}\rangle$$
$$\Leftrightarrow |z|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2p\langle \mathfrak{a},\mathfrak{b}\rangle \geq \|\mathfrak{b}\|\,\|\mathfrak{a}\|^2 + \|\mathfrak{b}\|^2 - 2p\langle \mathfrak{a},\mathfrak{b}\rangle$$
$$\Leftrightarrow \|z\mathfrak{a} - \mathfrak{b}\| \geq \|\mathfrak{b}\|\,\|\mathfrak{a}\| \qquad (25)$$

Consequently, $z \in \Gamma(\mathfrak{R})$ and we can choose $C(z) = \|\mathfrak{b}\|$.

Since $z \in \Gamma(\mathfrak{R})$, we have $(z - \mathfrak{R})^{-1} \in \mathcal{B}(\mathfrak{H})$ and $(z - \mathfrak{R})$ is an injective.

This prove that $z \in \rho(\mathfrak{R})$.

Corollary 12 If relation $\mathfrak{R}$ is closed and bounded and $|z| \leq \|\mathfrak{b}\|$, then $z \in \sigma(\mathfrak{R})$.

CONCLUSION

A linear relation $\mathfrak{R}$ on $\mathfrak{H}$ is a set of pairs $\mathfrak{a}$ and $\mathfrak{b}$ elements with $\mathfrak{a},\mathfrak{b} \in \mathfrak{H}$ and denoted $\mathfrak{R} = \{(\mathfrak{a},\mathfrak{b}) : \mathfrak{a},\mathfrak{b} \in \mathfrak{H}\}$. We give some characteristics of linear relation that if $\mathfrak{R}$ is an injective relation on $\mathfrak{H}$ and range of $\mathfrak{R}$ is dense then $\mathfrak{R}^*$ is also injective and $(\mathfrak{R}^*)^{-1} = (\mathfrak{R}^{-1})^*$. Consequently, if a relation $\mathfrak{R}$ on $\mathfrak{H}$ is an injective and self-adjoint, then relation $\mathfrak{R}^{-1}$ is a self-adjoint. If a symmetric relation $\mathfrak{R}$ on $\mathfrak{H}$ then $M(z-\mathfrak{R}^*) = R(z-\mathfrak{R})^{\perp} \cap D(z-\mathfrak{R}^*)$ and $M(z-\mathfrak{R}) = R(z-\mathfrak{R}^*)^{\perp} \cap D(z-\mathfrak{R})$. We give the results of spectral analysis that if a relation $\mathfrak{R}$ is a symmetric and $z \in \mathbb{C}$, then $\|(z\mathfrak{a} - \mathfrak{b})\|^2 \geq q^2\|\mathfrak{a}\|^2$. If a relation $\mathfrak{R}$ is closed and bounded and $|z| \geq \|\mathfrak{b}\|$, then $z \in \rho(\mathfrak{R})$. Consequently, if a relation $\mathfrak{R}$ is closed and bounded and $|z| \leq \|\mathfrak{b}\|$, then $z \in \sigma(\mathfrak{R})$.

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