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Submission date: 13-Apr-2023 11:48AM (UTC+0700)
Submission ID: 2063209631
File name: olikhin_Primitive_Function_of_the_Dunford_Integral_organized.pdf (505.02K)
Word count: 3506
Character count: 14475

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Abstract. In this paper, Dunford integrals and their primitive functions are discussed. We discuss its properties related to absolutely continuous, strictly absolutely continuous, bounded variation function, strictly bounded variation function and their generalizations. The result is obtained that for each function which integrated Dunford, then the primitive function is a continuous, absolutely continuous, and bounded variation function. Furthermore, its generalized absolutely continuous and generalized bounded variation function.

1. Introduction

Newton has define the integral of the function f from $[a,b]$ to R through anti-derivatives F on $[a,b]$, with $F'(x)=f(x)$ for every x element $[a,b]$. A function F is called primitive of f on $[a,b]$ and $\int_a^b f(x)dx = F(b)-F(a)$ [1]. While, Riemann has define the integral of function f through a function constant $\delta > 0$, i.e. $f:[a,b]\to R$ is said to be Riemann integrable to R_R on $[a,b]$ if for every $\varepsilon > 0$ there is constant $\delta > 0$ such that for any partition $P=\{x_0,x_1,...,x_n\}$ on $[a,b]$ we have

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - R_R\right| < \varepsilon .$$

Based on properties of Riemann integral, we defined primitive the based F on $[a,b]$ by $F(x)=(R_R)\int_a^x f(\alpha)d\alpha$ and if f is continuous, then F is anti-derivatives. In addition of Mathematics, the Riemann integral is used in physics and engineering. The development of the types of function in physics and engineering gives an impact on the function that is not Riemann integrable.

Lebesgue resolves issues that the Reimann integral has not resolved. Lebesgue has define the integral of the function $f:[a,b]\to R$ through measure [2]. A function f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable. One of the applications of Lebesgue integral is determining the coefficients of the Fourier formula [3].Space of Lebesgue integrable function is complete normed space. This has caught attention of researchers.

Based on the Lebesgue integral, the integral of weakly measurable function f is Banach-valued function like Donford said. A function $f:[a,b]\to X$ be Dunford integrable on $[a,b]$ if for each $x^* \in X^*$

real-valued function $x^*(f)$ is Lebesgue integrable [3]. Some properties of Dunford integral has discussed. Linear space and seminorm space be the space of Dunford integrable functions. Furthermore, operators which work on space of Dunford integrable function is linear and bounded operators [4]. It is weakly compact linear operators [4], [5].

From Lebesgue integral and Dunford integral, Dunford integrals and their primitive functions are discussed. We discuss its properties related to absolutely continuous, strictly absolutely continuous, bounded variation function, strictly bounded variation function and their generalizations.In this article, nothing new is found. We collaborate from something that already exists [1], [2], [4].

2. Absolutely Continuous Functions and Functions of Bounded Variation and their Generalization

The following discusses among absolutely continuous function, function of bounded variation, and their generalization. We show their relation.

Definition 2.1 Let function $F:[a,b]\to X$ *with* X *is Banach space and* $A\subset[a,b]$. *A function* F *is called absolutely continuous on* A, *denote* $F\in AC(A)$, *if for every* $\varepsilon>0$ *there is a* $\delta>0$ *such that* $\{(x_i,y_i)\}$ *is sequence of non overlapping intervals on* $[a,b]$, $x_i,y_i\in A$ *and* $\sum_{i=1}^{\infty}(y_i-x_i)<\delta$ *implies*

$$\sum_{i=1}^{\infty}\|F(x_i,y_i)\|_X=\sum_{i=1}^{\infty}\|F(y_i)-F(x_i)\|_X<\varepsilon .$$

Definition 2.2 *A function* F *is called restricly absolutely continuous on* A, *denote* $F\in AC^*(A)$, *if for every* $\varepsilon>0$ *there is a* $\delta>0$ *such that* $\{(x_i,y_i)\}$ *is sequence of non overlapping intervals on* $[a,b]$, $x_i,y_i\in A$ *and* $\sum_{i=1}^{\infty}(y_i-x_i)<\delta$ *implies*

$$\sum_{i=1}^{\infty}\omega(F;[x_i,y_i])<\varepsilon$$

where $\omega(F;[x_i,y_i])=\sup\{\|F(x)-F(y)\|_X : x,y\in[x_i,y_i], i\geq 1\}$.

By Definition 2.1 and Definition 2.2 we easily proof, if $F\in AC^*(A)$ then $F\in AC(A)$ and $AC^*(A)\subset AC(A)$.

Theorem 2.3 *Let* $A\subset[a,b]$ *is closed set,* $(a,b)-A=\bigcup_{k=1}^{\infty}(c_k,d_k)$ *and* F *is continuous for every* $x\in[a,b]$. *The following conditions are equivalent:*

(i). F *is restricly absolutely continuous on* A,

(ii). F *is absolutely continuous on* A *and*

$$\sum_{i=1}^{\infty}\omega(F;[c_i,d_i])<\infty .$$

(iii).For every $\varepsilon>0$ *there is a* $\delta>0$ *such that* $\{(a_i,b_i)\}$ *is sequence of non overlapping intervals with*

$a_i\in A$ *or* $b_i\in A$ *and* $\sum_{i=1}^{\infty}(b_i-a_i)<\delta$ *we have*

$$\sum_{i=1}^{\infty}\left\|F(b_i)-F(a_i)\right\|_X<\varepsilon . \qquad \square$$

Definition 2.4 *A function* F *is called bounded variation on* $A\subset[a,b]$, *denote* $F\in BV(A)$, *if there is a* $M\geq 0$ *such that for every sequence of non overlapping intervals* $\{(a_j,b_j)\}$, $a_j,b_j\in A$ *we have*

$$\sum_{j=1}^{\infty}\left\|F(b_j)-F(a_j)\right\|_X\leq M .$$

Definition 2.5 *A function* F *is called restricly bounded variation on* $A\subset[a,b]$, *denote* $F\in BV^*(A)$, *if there is a* $M\geq 0$ *such that for every sequence of non overlapping intervals* $\{(a_j,b_j)\}$, $a_j,b_j\in A$ *we have*

$$\sum_{j=1}^{\infty}\omega(F;[a_j,b_j])\leq M .$$

The relation between AC and BV.

Theorem 2.6 *If* $F\in AC(A)$ *then* $F\in BV(A)$.

Proof. by Definition 2.1 and Definition 2.4. $\square$

Theorem 2.7 *If* $F\in AC^*(A)$ *then* $F\in BV^*(A)$.

Proof. We known $F\in AC^*(A)$. Its means for every $\varepsilon>0$ (fix $\varepsilon=1$) there is a $\delta>0$ such that for every $\{(c_i,d_i)\}$ on $[a,b]$, $c_i,d_i\in A$ and $\sum_{i=1}^{\infty}(d_i-c_i)<\delta$ we have

$$\sum_{i=1}^{\infty}\omega(F;[c_i,d_i])<\varepsilon=1 .$$

Takes $M=\sup\left\{\sum_{i=1}^{\infty}\omega(F;[c_i,d_i])<1 : c_i,d_i\in A\right\}$.

Then there is a $M\geq 0$ such that for every $\{(a_i,b_i)\}$, $a_i,b_i\in A$ we have

$$\sum_{i=1}^{\infty}\omega(F;[a_i,b_i])\leq M .$$

So, $F\in BV^*(A)$. $\square$

Theorem 2.8 *If* $F\in BV^*(A)$ *then* $F\in BV(A)$.

Proof. $F\in BV^*(A)$ i.e. there is a $M\geq 0$ such that for every $\{(a_i,b_i)\}$, $a_i,b_i\in A$ we have

$$\sum_{i=1}^{\infty}\omega(F;[a_i,b_i])\leq M .$$

It is implies

$$\sum_{i=1}^{\infty}\left\|F(a_i,b_i)\right\|_X=\sum_{i=1}^{\infty}\left\|F(b_i)-F(a_i)\right\|_X$$

$$\leq \sum_{i=1}^{\infty} \sup\{\|F(x)-F(y)\|_X : x,y \in [a_i,b_i], i \geq 1\}$$
$$= \sum_{i=1}^{\infty} \omega(F;[a_i,b_i])$$
$$\leq M .$$

In other word $F \in BV(A)$. □

Definition 2.9 *Let* X *is Banach space and function* $F:[a,b] \to X$. *A function* F *is called generalized absolutely continuous function on* $A \subset [a,b]$, *denote* $F \in ACG(A)$, *if there exists sequence* $\{a_n\}$ *such that*

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in AC(a_n), \forall n .$$

Definition 2.10 *A function* F *is called generalized stricly absolutely continuous function on* $A \subset [a,b]$ *, denote* $F \in ACG^*(A)$, *if there exists sequence* $\{a_n\}$ *such that*

$$A = \bigcup_{n=1}^{\infty} a_n \text{ dan } F \in AC^*(a_n), \forall n .$$

Theorem 2.11 *If* $F \in AC(A)$ *then* $F \in ACG(A)$.

Proof. $F \in AC(A)$.

We construct a sequence $\{a_n\}$ such that $A = \bigcup_{n=1}^{\infty} a_n$. We known $F \in AC(A)$ and $A = \bigcup_{n=1}^{\infty} a_n$ then $F \in AC\left(\bigcup_{n=1}^{\infty} a_n\right)$. It implies $F \in AC(a_n)$, $\forall n$.

Hence, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in AC(a_n), \forall n .$$

So, $F \in ACG(A)$. □

Theorem 2.12 *If* $F \in AC^*(A)$ *then* $F \in ACG^*(A)$.

Proof. $F \in AC^*(A)$. We construct a sequence $\{a_n\}$ such that $A = \bigcup_{n=1}^{\infty} a_n$. In fact $F \in AC^*(A)$ and $A = \bigcup_{n=1}^{\infty} a_n$ then $F \in AC^*\left(\bigcup_{n=1}^{\infty} a_n\right)$. It implies $F \in AC^*(a_n)$, $\forall n$.

Therefore, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in AC^*(a_n), \forall n .$$

So, $F \in ACG^*(A)$. □

Theorem 2.13 *If* $F \in ACG^*(A)$ *then* $F \in ACG(A)$.

Proof. $F \in ACG^*(A)$ i.e. there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in AC^*(a_n), \forall n.$$

In fact, if $F \in AC^*(a_n), \forall n$ then $F \in AC(a_n), \forall n$.

Hence, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in AC(a_n), \forall n.$$

In other word $F \in ACG(A)$. □

Definition2.14 *A function F is called generalized bounded variation function on* $A \subset [a,b]$, *denote* $F \in BVG(A)$, *if there exists sequence* $\{a_n\}$ *such that*

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV(a_n), \forall n.$$

Definition 2.15 *A function F is called function of generalized stricly bounded variation on* $A \subset [a,b]$, *denote* $F \in BVG^*(A)$, *if there exists sequence* $\{a_n\}$ *such that*

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV^*(a_n), \forall n.$$

Theorem 2.16 *If* $F \in BV(A)$ *then* $F \in BVG(A)$.

Proof. We know that $F \in BV(A)$, its means there is a $M \geq 0$ such that for every $\{(a_i, b_i)\}$, $a_i, b_i \in A$,

$$\sum_{i=1}^{\infty} \|F(a_i, b_i)\|_X \leq M.$$

We construct a sequence $\{a_n\}$ such that $A = \bigcup_{n=1}^{\infty} a_n$. We known $F \in BV(A)$ and $A = \bigcup_{n=1}^{\infty} a_n$ then $F \in BV\left(\bigcup_{n=1}^{\infty} a_n\right)$. We obtained $F \in BV(a_n), \forall n$.

Hence, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV(a_n), \forall n.$$

So, $F \in BVG(A)$. □

Theorem 2.17 *Jika* $F \in BV^*(A)$ *maka* $F \in BVG^*(A)$.

Journal of Physics: Conference Series **1943** (2021) 012134 doi:10.1088/1742-6596/1943/1/012134

Proof. We known $F \in BV^*(A)$. We construct a sequence $\{a_n\}$ such that $A = \bigcup_{n=1}^{\infty} a_n$. In fact $F \in BV^*(A)$ and $A = \bigcup_{n=1}^{\infty} a_n$ then $F \in BV^*\left(\bigcup_{n=1}^{\infty} a_n\right)$. Its imples $F \in BV^*(a_n),\ \forall n$.

It yields, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV^*(a_n),\ \forall n.$$

Now we have concluded that $F \in BVG^*(A)$. □

Theorem 2.18 *If* $F \in BVG^*(A)$ *then* $F \in BVG(A)$.

Proof. $F \in BVG^*(A)$ i.e. there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV^*(a_n), \forall n.$$

By theorems, $F \in BV^*(a_n), \forall n$ then $F \in BV(a_n), \forall n$.

It means, there exists sequence $\{a_n\}$ such that

$$A = \bigcup_{n=1}^{\infty} a_n \text{ and } F \in BV(a_n), \forall n.$$

In other side, $F \in BVG(A)$. □

We conclude $AC^*(A) \subset ACG^*(A) \subset ACG(A)$; $AC^*(A) \subset AC(A) \subset ACG(A)$ $AC^*(A) \subset BV^*(A) \subset BVG^*(A) \subset BVG$; or $AC^*(A) \subset BV^*(A) \subset BV(A) \subset BVG$.

3. Primitive Function of the Dunford integral

Before, we defined primitive function of the Dunford integral and relation among primitive, absolutely continuous, bounded variation and their generalized. We will define Dunford integral. As following.

Definition 3.1 [5] *Let* X *be Banach space and* X^* *be dual of* X. *A function* $f:[a,b] \to X$ *be Dunford integrable on* $[a,b]$ *,denote* $f \in D_L[a,b]$, *if for each* $x^* \in X^*$, $x^*(f):[a,b] \to R$ *is Lebesgue integrable and for each* $A \subset [a,b]$ *set of measurable there is a* $x^{**}_{(f,A)} \in X^{**}$ *we have*

$$x^{**}_{(f,A)}(x^*) = \int_A x^*(f),$$

for every $x^* \in X^*$.

Denote $D_L[a,b]$ is set of all function which it is Dunford integrable.

Theorem 3.2 [5] *The function* f *be Dunford integrable on* $[a,b]$ *if and only if for each* $x^* \in X^*$ $x^*(f)$ *be Lebesgue integrable on* $[a,b]$.

Proof. It is obviously by Definition 3.1

Now, we define primitive function of the Dunford integral.

Journal of Physics: Conference Series **1943** (2021) 012134 doi:10.1088/1742-6596/1943/1/012134

Definition 3.3 *Let* $\mathcal{I}[a,b]$ *be collection of all closed interval in* $[a,b]$, *and* $f:[a,b]\to X$. *If* $f\in D_L[a,b]$ *then* $F:\mathcal{I}[a,b]\to X$ *by*

$$F(A)=x^{**}_{(f,A)}=(D_L)\int_A f$$

and $F(\varnothing)=0$ *for every* $A\in\mathcal{I}[a,b]$ *is called primitives Dunford of* f *on* $[a,b]$.

Example 3.4 We defined function $f:[a,b]\to X$ *by*

$$f(x)=c\ ,$$

for every $x\in[a,b]$ and some constant $c\in X$, then for any closed intervals $A\subset[a,b]$ primitives Dunford of f is $F(A)=c\alpha(A)$.

Definition 3.5 *A function* $F:\mathcal{I}[a,b]\to X$ *is called additive, if*

$$F(P\cup Q)=F(P)+F(Q)$$

for each $P,Q\in\mathcal{I}[a,b]$ *where* $P\cup Q\in\mathcal{I}[a,b]$ and $P\cap Q$ *is empty set.*

Theorem 3.6 *If* $f\in D_L[a,b]$ *with the primitive F, then F is additive on* $[a,b]$.

Proof. given $A\subset[a,b]$ and $B\subset[a,b]$ respectively any closed intervals, $A\cup B\subset[a,b]$ and non overlapping, then for each $x^*\in X^*$ x^*f is Lebesgue integrable on $[a,b]$ and there is a $x^{**}_{(f,A)}\in X^{**}$ and $x^{**}_{(f,B)}\in X^{**}$ such that

$$x^{**}_{(f,A)}(x^*)=(L)\int_A x^*f \text{ and}$$

$$x^{**}_{(f,B)}(x^*)=(L)\int_B x^*f\ .$$

Therefore, there is a $x^{**}_{(f,A\cup B)}\in X^{**}$ such that

$$x^{**}_{(f,A)}(x^*)+x^{**}_{(f,B)}(x^*)=(L)\int_A x^*(f)+(L)\int_B x^*(f)=(L)\int_{A\cup B} x^*(f)=x^{**}_{(f,A\cup B)}(x^*)$$

So,

$$x^{**}_{(f,A\cup B)}=x^{**}_{(f,A)}+x^{**}_{(f,B)} \text{ or } F(A\cup B)=F(A)+F(B).$$ □

Corollary3.7 *If* $f\in D_L[a,b]$ *with the primitive* F *and* $A_1,A_2,...,A_n$ *respectively any closed intervals in* $[a,b]$ *which non overlapping and* $\bigcup_{i=1}^{n}A_i=[a,b]$ *, then*

$$F\left(\bigcup_{i=1}^{n}A_i\right)=\sum_{i=1}^{n}F(A_i)=\sum_{i=1}^{n}x^{**}_{(f,A_i)}$$

Proof: We known $f\in D_L[a,b]$ and F is primitive of f , $\bigcup_{i=1}^{n}A_i=[a,b]$ where $\mu_\alpha(A_i\cap A_j)=0$ for every $i\neq j$ we have

$$F\left(\bigcup_{i=1}^{n}A_i\right)=x^{**}_{\left(f,\bigcup_{i=1}^{n}A_i\right)}=x^{**}_{(f,A_1)}+x^{**}_{(f,A_2)}+...+x^{**}_{(f,A_n)}=F(A_1)+F(A_2)+...+F(A_n)$$

$$= \sum_{i=1}^{n} F(A_i) = \sum_{i=1}^{n} x^{**}_{(f,A_i)} . \qquad \square$$

Next, by Definition 3.1, Definition 3.3 and equivalent of Lebesgue integral and McShane integral, we can define Dunford integral as in the following theorems.

Theorem 3.8 *A function $f \in D_L[a,b]$ iff there exists additive function F on $[a,b]$ such that for every $\varepsilon > 0$ and $x^* \in X^*$ there is a positive function δ on $[a,b]$ and for any $A \subset [a,b]$ closed intervals and McShane partition $\mathcal{D} = \{(D_1,x_1),(D_2,x_2),...,(D_n,x_n)\}$ δ-fine on A we have*

$$\left| \sum_{i=1}^{n} x^* \left(f(x_i)\alpha(D_i) - F(D_i) \right) \right| < \varepsilon$$

or

$$\left| \sum_{i=1}^{n} x^* f(x_i)\alpha(D_i) - x^* F(A) \right| < \varepsilon \ . \square$$

Theorem 3.9 (Sack-Henstock Lemma) *A function $f \in D_L[a,b]$ and F its primitive, i.e. for every $\varepsilon > 0$ and x^* element X^* there is a $\delta > 0$ on $[a,b]$ such that for any $A \subset [a,b]$ closed intervals and McShane partitions $\mathcal{D} = \{(D,x)\}$ δ-fine on A we have*

$$\left| \mathcal{D}\sum x^* f(x)\alpha(D) - x^* F(D) \right| < \varepsilon$$

Then for any partial sums $\sum_1$ of $\mathcal{D}\sum$ we have

$$\left| \mathcal{D}\sum_1 x^* f(x)\alpha(D) - x^* F(D) \right| < \varepsilon . \qquad \square$$

Theorem 3.10 *If $f \in D_L[a,b]$ with the primitive F, then F be continuous on $[a,b]$.*

Proof.by Sack-Henstock Lemma and inequality, easy to proof this theorem.

Theorem 3.11 *If $f \in D_L[a,b]$ and F its primitive, then $F \in BVG[a,b]$.*

Proof.we known $f \in D_L[a,b]$ i.e. for every $\varepsilon > 0$ and $x^* \in X^*$ there is a ν on $[a,b]$ and for any $A \subset [a,b]$ closed intervals and McShane partition $\mathcal{D} = \{(D_1,x_1),(D_2,x_2),...,(D_n,x_n)\}$ δ-fine on A we have

$$\left| \sum_{i=1}^{n} x^* \left(f(x_i)\alpha(D_i) - F(D_i) \right) \right| < \varepsilon$$

or

$$\left| \sum_{i=1}^{n} x^* f(x_i)\alpha(D_i) - x^* F(A) \right| < \varepsilon \ .$$

Given fix $\varepsilon = 1$, $\delta(x) \le 1$ with $\|x^*\|_X \le 1$.

We construct set A_{ni} is sets of all $x \in [a + \frac{i-1}{n}, a + \frac{i}{n}] \cap [a,b]$ with $\|f(x)\|_X \le n$ and $\frac{1}{n} < \delta(x) \le \frac{1}{n-1}$

.

We obtained $[a,b]=\bigcup_{\substack{i=1\\n=2}}^{\infty} A_{ni}$.

Given arbitrary sets of closed intervals non overlapping $\{[a_k,b_k]\}$, $a_k,b_k \in A_{ni}$ for all k .

We obtained $\{(a_k,[a_k,b_k])\}$ is Perron partition δ-fine $K-$system in $[a,b]$.

Hence, we have

$$\left\|\sum_{k=1}^{\infty} F([a_k,b_k])\right\| \le \|x^*\|_X \left\|\sum_{k=1}^{\infty} F([a_k,b_k])\right\|_X$$

$$\le \|x^*\|_X \left\|\sum_{k=1}^{\infty} F([a_k,b_k]) - f(a_k)(b_k-a_k)+f(a_k)(b_k-a_k)\right\|_X$$

$$\le \|x^*\|_X \left\|\sum_{k=1}^{\infty} F([a_k,b_k]) - f(a_k)(b_k-a_k)\right\|_X + \|x^*\|_X \left\|\sum_{k=1}^{\infty} f(a_k)(b_k-a_k)\right\|_X$$

$$< 1+n(b-a).$$

It is shown that $F \in BV(A_{ni})$.

So, $F \in BVG[a,b]$. □

Corollary 3.12 *If* $f \in D_L[a,b]$ *and* F *its primitive, then there exists* $\{A_{ni}\}$ *is sequence of sets such that* $[a,b]=\bigcup_{\substack{i=1\\n=2}}^{\infty} A_{ni}$ *and* $F \in BV(A_{ni}), \forall n,i$. □

Theorem 3.13 *If* $f \in D[a,b]$ *with the primitive* F *, then* $F \in BV[a,b]$.

Proof. We known $f \in D_L[a,b]$ with the primitive F , by Corollary 3.12 there exists sequence of sets $\{A_{ni}\}$ such that $\bigcup_{\substack{i=1\\n=2}}^{\infty} A_{ni}=[a,b]$ and $F \in BV(A_{ni}), \forall n,i$. $F \in BV(A_{ni}), \forall n,i$ and $[a,b]=\bigcup_{\substack{i=1\\n=2}}^{\infty} A_{ni}$, implies $F \in BV[a,b]$. □

Theorem 3.14 *If* $f \in D_L[a,b]$ *and* F *its primitive, then* $F \in ACG[a,b]$.

Proof. we know $f \in D_L[a,b]$. Its means for every $\varepsilon>0$ and $x^* \in X^*$ there is a $\delta>0$ on $[a,b]$ and for any $A \subset [a,b]$ closed interval and McShane partitions $\mathcal{D}=\{(D_1,x_1),(D_2,x_2),\dots,(D_n,x_n)\}$ $\delta-$fine on A we have

$$\left|\sum_{i=1}^{n} x^*\left(f(x_i)\alpha(D_i)-F(D_i)\right)\right| < \frac{\varepsilon}{2}$$

or

$$\left|\sum_{i=1}^{n} x^* f(x_i)\alpha(D_i) - x^* F(A)\right| < \frac{\varepsilon}{2} .$$

Suppose $\delta(x) \le 1$ and $\|x^*\|_X \le 1$.

We construct sets A_{ni} is sets of all point $x \in [a+\frac{i-1}{n}, a+\frac{i}{n}] \cap [a,b]$ such that $\|f(x)\|_X \le n$ and $\frac{1}{n} < \delta(x) \le \frac{1}{n-1}$.

We obtain $[a,b] = \bigcup_{\substack{i=1 \\ n=2}}^{\infty} A_{ni}$.

Given arbitrary sets of closed intervals non overlapping $\{[a_j,b_j]\}$ with $a_j, b_j \in A_{ni}$ for all j .

We have $\{(a_j,[a_j,b_j])\}$ is Perron partitions δ -fine $K-$ system in $[a,b]$, which implies $\{(a_j,[a_j,b_j])\}$ is McShane partitions δ -fine $K-$system in $[a,b]$.

Therefore,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} F([a_j,b_j])\right\| &\le \|x^*\|_X \left\|\sum_{j=1}^{\infty} F([a_j,b_j])\right\|_X \\
&\le \|x^*\|_X \left\|\sum_{j=1}^{\infty} F([a_j,b_j]) - f(a_j)(b_j-a_j) + f(a_j)(b_j-a_j)\right\|_X \\
&\le \|x^*\|_X \left\|\sum_{j=1}^{\infty} F([a_j,b_j]) - f(a_j)(b_j-a_j)\right\|_X + \|x^*\|_X \left\|\sum_{j=1}^{\infty} f(a_j)(b_j-a_j)\right\|_X \\
&\le \left|\sum_{j=1}^{\infty} x^* F([a_j,b_j]) - x^* f(a_j)(b_j-a_j)\right| + \|x^*\|_X \left\|\sum_{j=1}^{\infty} f(a_j)(b_j-a_j)\right\|_X \\
&< \frac{\varepsilon}{2} + n\sum_j (b_j-a_j).
\end{aligned}$$

Choose $\eta \le \frac{\varepsilon}{2n(b-a)}$ and $\sum_j (b_j-a_j) < \eta$, we obtained

$$\begin{aligned}
\left\|\sum_{k=1}^{\infty} F([a_k,b_k])\right\| &< \frac{\varepsilon}{2} + n(b-a)\eta \\
&\le \varepsilon .
\end{aligned}$$

Its means $F \in AC(A_{ni})$.

Hence, there exists sequence of sets $\{A_{ni}\}$ such that $[a,b] = \bigcup_{\substack{i=1 \\ n=2}}^{\infty} A_{ni}$ and $F \in AC(A_{ni})$, $\forall n,i$.

In other word $F \in ACG[a,b]$. □

Corollary 3.15 *If* $f \in D_L[a,b]$ *and F its primitive, then there exists* $\{A_{ni}\}$ *sequence of sets such that* $[a,b] = \bigcup_{\substack{i=1 \\ n=2}}^{\infty} A_{ni}$ *and* $F \in AC(A_{ni}), \forall n,i$. □

Theorem 3.16 *If* $f \in D_L[a,b]$ *and F its primitive, then* $F \in AC[a,b]$.

Proof. $f \in D[a,b]$ and F its primitive, by Corollary 3.15 there exists $\{A_{ni}\}$ is sequence of sets such that $[a,b]=\bigcup_{\substack{i=1\\n=2}}^{\infty} A_{ni}$ and $F \in AC(A_{ni}), \forall n,i$. So, $F \in AC[a,b]$. □

4. Conclusion

We have concluded that each function which Dunford integrable, then the primitive function be continuous, absolutely continuous, and bounded variation. Furthermore,the primitive function be generalized absolutely continuous and generalized bounded variation.

Acknowledgments

This research has been funded by FSM Undip with no. 1958/UN7.5.8/PP/2020. So that, the author say thank you.

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