Weakly compact linear operators on space of Dunford integral function

by Susilo Hariyanto

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Abstract. This study discussed the integral of Dunford and compact linear operator on space of Dunford integral function. For each f which is Dunford integral on $[a,b]$ is defined as an operator D_L by $D_L(x^*) = x^* f$, for each $x^* \in X^*$. This study resulted that the operator D_L is both a continuous linear operator and weakly compact operators. Then, it was defined as the adjoint of the operator D_L^* by $D_L^*(h)(x^*) = \int_a^b hD_L(x^*)$ each $h \in (L_1)^*$. The adjoint operator D_L^* is continuous and weakly compact linear operators.

1. Introduction

In many applications, for example in differential equations, optimization and so forth do not rule out the possibility of integral problems faced with Banach X valued functions. The study of integral theory for Banach's valued functions has developed and it becomes an interesting topic for researchers. Many researchers study the integral theory of a Banach-valued or vector-valued function which is the development of a real-valued function. For example, such as the Bochner integral [1], the Henstock-Bochner integral [2], the Henstock-Kurzweil integral and the McShane integral of Banach space-valued function [3], the Henstock-Pettis integral of Banach space-valued function [4], and Dunford integrals [5].

Integral Bochner was introduced by Salomon Bochner. He extended the Lebesgue integral definition [6] into the Banach valued function. The function f of a closed interval I into the Banach space X is Bochner integrable if there is a sequence of simple functions $(f_n), \forall n \in N$ such that $\lim_{n\to\infty} \int_I \|f_n - f\| d\mu = 0$ [5]. As for the Dunford integral, it defines the integral of a weakly measurable function. The weakly measurable function f is called Dunford integrable if for each $x^* \in X^*$ (X^* is the dual space of the Banach space X) real-valued function $x^* f$ is Lebesgue integrable [5].

Studies for Dunford integrals have been expanded into Riemann type integrals, such as the Henstock-Dunford integral and the Henstock-Pettis integral [7]. The integration of Henstock-Dunford requires that the function of real value $x^* f$ integral Henstock [8]. Integrated into Henstock[8]. Not only limited to this, but the Henstock-Dunford integral is also generalized in the Euclidean space, namely the Henstock-Dunford integral in the Euclidean space.

The study of integral theory is also combined with operator theory. Several properties of the positive Dunford-Pettis operator [9] have been discussed. Operators working in the space of Bochner integrable function. Next, operators are examined in the space of Dunford integrable function [10].

Taking into account the results of operator studies on space of Dunford integrable function. it will be examined more specifically about weakly compact operators and adjoint operators [11] on space of Dunford integrable function. How are the weakly compact operators and its adjoint operator and the relationship of the two operators.

2. Weakly measurable functions

The following discusses the measurable functions and weak measurable functions valued at Banach.

Definition 2.1 [5] *A function* $f:[a,b]\to X$ *is called simple , if there is a measurable set* $A_k \subset [a,b]$, $k=1,2,...,q$ *so that* $A_k \cap A_j = \varnothing$, *for each* $k \neq j$ *and* $[a,b]=\bigcup_{k=1}^{q} A_k$, *where* $f(x)=y_k \in X$ *for* $x \in A_k$, $k=1,2,...,q$.

Based on simple functions, measurable functions can be defined as follows.

Definition 2.2 [5] *A function f is said to be measurable if there is a simple sequence of functions* $(f_k), k\in N$ *with*

$$\lim_{k\to\infty}\left\|f_k(x)-f(x)\right\|_X=0,$$

for almost all $x\in[a,b]$.

According to Definition 2.2, so simple functions are measurable functions. Furthermore, if the f measured function , then the real function $\|f\|_X:[a,b]\to R$ is also a measurable function.

Theorem 2.3 [5] *If f is a measurable function , then the real function* $\|f\|_X$ *is a measurable function.*

Proof: Because the f functions are measurable, there are simple functions such $(f_k), k\in N$ so that

$$\lim_{k\to\infty}\left\|f_k(x)-f(x)\right\|_X=0$$

for each $x\in[a,b]$. Because f_k it is a simple function for all $k\in N$, therefore $\|f_k\|_X$ it is also a simple function for all $k\in N$. Therefore, f_k the function is measurable so it applies

$$\left|\left\|f_k(x)\right\|_X-\left\|f(x)\right\|_X\right|\le\left\|f_k(x)-f(x)\right\|_X,$$

for almost all $x\in[a,b]$ .This means $\lim_{k\to\infty}\left\|f_k(x)\right\|_X=\left\|f(x)\right\|_X$ that $\|f\|_X$ is measurable. □

Definition 2.4 [5] *A functions f said to be weakly measurable if for each* $x^*\in X^*$ *the real function* x^*f *is measurable*.

Each measurable function is a weakly measurable function

Theorem 2.5 [5] *If f is measurable, then f* it *is weakly measurable*.

Proof: Measured f function, meaning there is a sequence of simple functions (f_k), $k\in N$ so that

$$\lim_{k\to\infty}\left\|f_k(x)-f(x)\right\|_X=0,$$

for each $x\in[a,b]$.

Taken arbitrarily $x^*\in X^*$, then apply

$$\left|x^*\left(f_k(x)-f(x)\right)\right|\le\left\|x^*\right\|_X\left\|f_k(x)-f(x)\right\|_X,$$

and $\lim_{k\to\infty}\left|x^*\left(f_k(x)-f(x)\right)\right|=0$ .So, f is weakly measurable. □

Based on the measurable function of weak lemma construction which will guarantee the existence of the Dunford integral.

Lemma 2.6 Dunford [5] *Assume* X *is Banach space and* X^* *it is dual* X *. If* $f:[a,b]\to X$ *is a function that is weakly measurable and that for each* $x^*\in X^*$ *the function of real-value* $x^*f:[a,b]\to R$ *is Lebesgue integrable i.e.,* $x^*f\in L_1$ *, then for each measurable set* $A\subset[a,b]$ *there exists a unique vector* $x^{**}_{(f,A)}\in X^{**}$ *such that*

$$x^{**}_{(f,A)}\left(x^*\right)=\int_A x^*f,$$

for every $x^*\in X^*$ *.*

Proof: [5, 10] □

3. Dunford integral

The definition of the Dunford integral is given and it is shown that the set of all Dunford integrable functions is linear space.

Definition 3.1 [5] *Assume that* X *is Banach space and* X^* *it is dual of* X *. The function* $f:[a,b]\to X$ *said to be Dunford integrable on* $[a,b]$ *if for each* $x^*\in X^*$ *real functions* x^*f *is Lebesgue integrable and for each set of measurable* $A\subset[a,b]$ *there exists a unique vector* $x^{**}_{(f,A)}\in X^{**}$ *such that*

$$x^{**}_{(f,A)}\left(x^*\right)=\int_A x^*f,$$

for every $x^*\in X^*$ *.*

The set of all Dunford integrable functions is denoted by $D_L[a,b]$. For $f\in D_L[a,b]$ means that f is Dunford integrable on $[a,b]$.

Theorem 3.2 [10] *If* $f\in D_L[a,b]$*, then for every single measurable set* $A\subset[a,b]$ *vector* $x^{**}_{(f,A)}\in X^{**}$ *single.*

Proof: [10].

Examples of Dunford integrable functions are constant functions, continuous functions, Riemann integrable functions, Lebesgue integrable functions [10] and so on.

Theorem 3.3 [5] *The function* f *is Dunford integrable on* $[a,b]$ *if and only if for each* $x^*\in X^*$ *the function* x^*f *is Lebesgue integrable on* $[a,b]$ *.*

Proof: Based on Definition 3.1 if f is Dunford integrable on $[a,b]$, then for each $x^*\in X^*$ the function x^*f is Lebesgue integrable on $[a,b]$ .Conversely, if the real function x^*f is Lebesgue integrable on $[a,b]$, then f is Dunford integrable on $[a,b]$.

It was further shown that the collection of all functions f that Dunford integrated on $[a,b]$, is linear space.

Theorem 3.4 *The collection of all functions* f *that Dunford integrated on* $[a,b]$ *,* $D_L[a,b]$ *is linear space.*

Proof: Let $f,g\in D_L[a,b]$ arbitrarily $c\in R$. It shows that $f+g\in D_L[a,b]$ and $cf\in D_L[a,b]$. For $f,g\in D_L[a,b]$ then for each, $x^*\in X^*$ the function x^*f and x^*g are Lebesgue integrable on $[a,b]$. So that for any $x^*\in X^*$ the function $x^*\left(f+g\right)=x^*f+x^*g$ is Lebesgue integrable on $[a,b]$. So, the

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function $f+g$ is Dunford integrable on $[a,b]$. Furthermore, for any scalar $c \in R$ and for each $x^* \in X^*$ the function $x^*(cf) = cx^* f$ is Lebesgue integrable on $[a,b]$. So, the function cf is Dunford integrable on $[a,b]$.

4. Weakly compact operators on space of Dunford integral function

Suppose X is a Banach space with its dual X^* and X^{**} is dual X^*. By L_1 is space of Lebesgue integrable functions on $[a,b]$.

Defined operator $D_L : X^* \to L_1$ by

$$D_L(x^*) = x^* f,$$

for every $x^* \in X^*$.

The operator D_L as in the definition is a bounded linear or continuous linear operator.

Theorem 4.1 [5] *The operator* D_L *is a bounded linear operator*.

Proof: Let arbitrarily $x^*, z^* \in X^*$ and arbitrarily $c \in R$. We obtained

$$\begin{aligned} D_L(x^* + z^*) &= (x^* + z^*) f \\ &= x^* f + z^* f \\ &= D_L(x^*) + D_L(z^*), \end{aligned}$$

and

$$D_L(cx^*) = (cx^*) f = cx^* f = cD_L(x^*).$$

So it's a linear operator. Furthermore, according to the Closed Graph Banach Theorem, operators D_L are bounded. □

Lemma 4.2 [11], [12] *The dual space* L_1 *is* L_∞, *i.e.* $L_1^* = L_\infty$.

Proof: For each $y = \{y_n\} \in L_\infty$ functional formed f_y on L_1 by

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n .$$

We obtained that f_y linear and continuous, i.e. for any $x = \{x_n\}, z = \{z_n\} \in L_1$ and any scalar $c \in R$ gives

$$f_y(x+z) = \sum_{n=1}^{\infty} (x_n + z_n) y_n = f_y(x) + f_y(z) \text{ and } f_y(cx) = \sum_{n=1}^{\infty} (cx_n) y_n = cf_y(x).$$

Furthermore for each $x = \{x_n\} \in L_1$ gives

$$|f_y(x)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| \le \|x\|_1 \|y\|_\infty .$$

So that each $y = \{y_n\} \in L_\infty$ determines with a single functional linear continuous f_y at L_1, or $L_\infty \subset L_1^*$.

Instead, take any continuous functional linear f on L_1. For each, $x = \{x_n\} \in L_1$ it can be represented by

$$x = \sum_{n=1}^{\infty} x_n e_n ,$$

with $e_n = \{0, \dots, \underset{n}{1}, 0, \dots, 0\}$.

Since f is linear,

$$f(x)=f\left(\sum_{n=1}^{\infty}x_n e_n\right)=\sum_{n=1}^{\infty}x_n f(e_n).$$

This means that $f(x)$ it is a linear combination of rows of numbers $\{f(e_n)\}$ or continuous linear functional f depending on rows of numbers $\{f(e_n)\}$.

Furthermore, because the function f is continuous, so $|f(x)|$ bounded, that is $|f(x)|<\infty$. According to the Cauchy-Schwartz Inequality, it was obtained

$$|f(x)|=\left|f\left(\sum_{n=1}^{\infty}x_n e_n\right)\right|=\left|\sum_{n=1}^{\infty}x_n\left(f(e_n)\right)\right|\leq\|x\|_{L_1}\sup_{n\geq1}|f(e_n)|.$$

So that $f(x)$ bounded it must be $\sup_{n\geq1}|f(e_n)|<\infty$. This means that $y=\{f(e_n)\}\in L_\infty$.

Thus, any continuous linear function f in L_1 determining with a single vector $y=\{f(e_n)\}\in L_\infty$, then $L_1^*\subset L_\infty$.

So, that $L_\infty\subset L_1^*$ and $L_1^*\subset L_\infty$, so $L_1^*=L_\infty$. □

For each $f\in D_L[a,b]$ defined $D_L^*:L_1^*\to X^{**}$ by an operator

$$D_L^*(h)(x^*)=\int_a^b hD_L(x^*)=\int_a^b hx^*f,$$

for every $h\in L_1^*=L_\infty$.

An operator T^* is called adjoint operators against operators $T:X^*\to L_1$ on L_1.

Theorem 4.3 [5] *The adjoint operator* D_L^* *is bounded linear operators and*

$$\|D_L^*\|=\|D_L\|.$$

Proof: The operator adjoint D_L^* linear for arbitrary $h_1,h_2\in L_1^*=L_\infty$ and any scalars $c_1,c_2\in R$ gives

$$D_L^*(c_1h_1+c_2h_2)(x^*)=\int_a^b(c_1h_1+c_2h_2)D_L(x^*)=c_1\int_a^b\left(h_1D_L(x^*)\right)+c_2\int_a^b\left(h_2D_L(x^*)\right)$$
$$=c_1D_L^*(h_1)(x^*)+c_2D_L^*(h_2)(x^*).$$

Because $f\in D_L[a,b]$ and $f=D_L^*(h)$, $\|f\|\leq\|h\|\|D_L\|$ then $\|D_L^*(h)\|=\|f\|\leq\|h\|\|D_L\|$.

So,

$$\|D_L^*\|\leq\|D_L\|.$$

For each $x_0^*\neq\theta\in X^*$ there is $h_0\in X^{**}$ with $\|h_0\|=1$ and $h_0\left(D_L(x_0^*)\right)=\|D_L(x_0^*)\|$.

So,

$$h_0\left(D_L(x_0^*)\right)=D_L^*(h_0)(x_0^*).$$

Let $f_0=D_L^*(h_0)$ and obtained

$$\|D_L(x_0^*)\|=h_0D_L(x_0^*)=f_0(x_0^*)\leq\|f_0\|\|x_0^*\|=\|D_L^*(h_0)\|\|x_0^*\|\leq\|D_L^*\|\|h_0\|\|x_0^*\|.$$

Since $\|h_0\|=1$, so for every $x_0^*\neq\theta\in X^*$ such that $\|D_L(x_0^*)\|\leq\|D_L^*\|\|x_0^*\|$.

So,

$$\|D_L\|\leq\|D_L^*\|.$$

Because $\|D_L\| \le \|D_L^*\|$ and $\|D_L^*\| \le \|D_L\|$, then $\|D_L^*\| = \|D_L\|$.

Theorem 4.4 *If* $D_L : X^* \to L_1$ *and* $T_L : X^* \to L_1$ *are bounded linear operators and any scalar* $c \in R$, *then*

(i) $(D_L + T_L)^* = D_L^* + T_L^*$.

(ii) $(cD_L)^* = cD_L^*$.

(iii) $\|D_L D_L^*\| = \|D_L^* D_L\| = \|D_L\|^2 = \|D_L^*\|^2$.

Proof: (i) $(D_L + T_L)^*(h)(x^*) = \int_a^b h(D_L + T_L)(x^*) = \int_a^b hD_L(x^*) + \int_a^b hT_L(x^*) = D_L^*(h)(x^*) + T_L^*(h)(x^*)$.

(ii) $(cD_L)^*(h)(x^*) = \int_a^b h(cD_L)(x^*) = c\int_a^b hD_L(x^*) = cD_L^*(h)(x^*)$.

(iii) Because D_L and D_L^* are bounded linear operators and $\|D_L^*\| = \|D_L\|$, then

$\|D_L D_L^*\| = \|D_L^* D_L\| = \|D_L\|^2 = \|D_L^*\|^2$. □

If $f \in D[a,b]$ the adjoint operator T^* is a weakly compact operator.

Theorem 4.5 [5] *Assume that* $f : [a,b] \to X$ *is Dunford integrable. The operator* $D_L : X^* \to L_1$ *is weakly compact operator if and only if the adjoint operator* $D_L^* : L_\infty \to X^{**}$ *is weakly compact operator.*

Proof: According to Gantmacher's theorem, the operator D_L is a weakly compact operator if and only if its adjoint operator D_L^* is also a weakly compact operator. □

5. Conclusion

Based on the results of the discussion outlined in the form of several theorems, it can be concluded that the collection of all functions integrated Dunford is linear space. For each function which is Dunford integral can be constructed by an operator. Its continuous linear operator and weakly compact operators. Furthermore, the adjoint operator is continuous and weakly compact linear operators.

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