Spectral theory for self-adjoint linear relation (SALR) on a Hilbert space and its application in homogenous abstract cauchy problem

by Susilo Hariyanto

Spectral theory for self-adjoint linear relation (SALR) on a Hilbert space and its application in homogenous abstract cauchy problem

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Abstract. A spectral theory studies eigenvalues and eigenvectors of SALR on H. SALR on Hilbert space H is a linear relation satisfying $\mathrm{A} = \mathrm{A}^*$. Many applications of SALR on quantum theory, such as the homogenous abstract Cauchy problem.If M is an operator that has an inverse then eigenvalues and eigenvectors are easily determined, but If M is an operator that does not have an inverse then eigenvalues and eigenvectors are quite difficult determined. One way that can be done is to use a linear relation. Furthermore, there are some properties of spectral theoryof linear operator that can not apply to SALR. This paper aims to give a spectral theory for SALR and its application in a homogenous abstract Cauchy problem.

1. Introduction

A linear relation is generally referred to as multivalued linear operator. A linear relation on Hilbert space H is a subspace of Hilbert space $\mathrm{H} \oplus \mathrm{H}$ [1]. The research about spectral theory of linear relation can be found [1-8]. Arens [1]analyzed spectral theory of SALR considered with an unitary operator through Cayley transformation. Gheorge and Vasilescu [2]founded strong connection between spectral theory of closed linear relation and closed linear operator. Recently, Gheorge and Vasilescu [2] given some properties of closed linear relation. Langer and Textorious [3] analyzed spectral theory of a linear relation associated with minimal self-adjont extension. A spectral theory of linear relation and its application can be found in Baskakov and Chernyshov [4], Baskokov and Zagorskii [5] and Sari, et al [6].

A linear relation has been used anabstract Cauchy problem. An abstract Cauchy problems are often found in chemistry, biology, physics, engineering, ecology, finance, industry, environment, and so on. Consider a homogenousabstract Cauchy problem on Hilbert space

$$\frac{d}{dt}Mr(t) = Lr(t), t \in \mathbb{R}_+ = [0, +\infty) \qquad (1)$$
$$r(0) = r_0$$

where M and L arelinear operator on Hilbert space H.Anabstract Cauchy problem is called degenerate if an operator M is not invertible. A Cauchy problem is called nondegenerate if an operator M is invertible. If the M operator has an inverse then the problem (1) can be written in the form:

$$\frac{d}{dt}Mr(t) = Lr(t), t \in \mathbb{R}_+ = [0,+\infty) \quad (2)$$
$$r(0) = r_0$$

where M and L are linear operator on Hilbert space H. Therefore, the determination of eigenvalues and eigenvectors on problem (2) is easy to do. If the operator M does not have an inverse then the problem (1) can be generalized in the form of a linear relation:

$$\frac{d}{dt}r(t) \in \mathrm{A}r(t), t \in \mathbb{R}_+ = [0,+\infty) \quad (3)$$
$$r(0) = r_0$$

where $\mathrm{A} = M^{-1}L$ is SALR on H. Wegive a spectral theory of SALR and its applications in homogenous abstract Cauchy problem.

2. Preliminaries

Some notations of linear relation on H can be seen in [1-4, 9-16]. A definition of linear relation, or relation for short, on Hilbert space H is as follow

Definition 1 [1] A linear relation is defined by $\mathrm{A} = \{(w,x) : w,x \in \mathsf{H}\}$. The domain of A is defined by $D(\mathrm{A}) = \{w : (w,x) \in \mathrm{A}\}$. The range of A is defined by $R(\mathrm{A}) = \{x : (w,x) \in \mathrm{A}\}$. The kernel of A is defined by $N(\mathrm{A}) = \{w : (w,0) \in \mathrm{A}\}$. The multivalued part of A is defined by $M(\mathrm{A}) = \{x : (0,x) \in \mathrm{A}\}$.

A identity relation on H is defined by $I = \{(w,w) : w \in \mathsf{H}\}$ and a zero relation is defined by $O = \{(w,0) : w \in \mathsf{H}\}$. The class of all linear relation on H will be denoted by $LR(\mathsf{H})$. An inverse relation on Hilbert space is defined by $\mathrm{A}^{-1} = \{(x,w),(w,x) \in \mathrm{A}\}$. Afterward, the duality of A and its inverse A^{-1} is

$$D(\mathrm{A}^{-1}) = R(\mathrm{A}), R(\mathrm{A}^{-1}) = D(\mathrm{A}), N(\mathrm{A}^{-1}) = M(\mathrm{A}), M(\mathrm{A}^{-1}) = N(\mathrm{A}).$$

The following is given the definition of an adjoint relation A^* on Hilbert space H.

Definition 2 [9] An adjoint relation of a relation A on H isa closed relation denoted by $\mathrm{A}^* = \{(k,l) \in \mathsf{H}^2 : \langle x,k \rangle = \langle w,l \rangle, \forall (w,x) \in \mathrm{A}\}$.

We give the following operations of relation.

Definition 3 [1] Let $\mathrm{A},\mathrm{B} \in LR(\mathsf{H})$, then the sum $\mathrm{A} + \mathrm{B}$ is defined by

$$\mathrm{A} + \mathrm{B} = \{(w, x+l) : (w,x) \in \mathrm{A}, (w,l) \in \mathrm{B}\}.$$

The product (composition) BA is defined by

$$\mathrm{BA} = \{(w,l) : \exists x \in \mathsf{H}, (w,x) \in \mathrm{A}, (x,l) \in \mathrm{B}\}.$$

A relation $z\mathrm{A}$ is defined by $z\mathrm{A} = \{(w,zx) : (w,x) \in \mathrm{A}\}$ for $z \in \mathbb{C}$. A relation $z - \mathrm{A}$ is defined by $z - \mathrm{A} = \{(w, zw - x) : (w,x) \in \mathrm{A}, z \in \mathbb{C}\}$.

The definition of symmetric, self adjoint, isometry, injective, and surjective relation is as follows.

Definition 4 [9] Let A isa relation on H, then

symmetric if $\mathrm{A} \subset \mathrm{A}^*$
self-adjoint if $\mathrm{A} = \mathrm{A}^*$
Isometri if $\langle w,k \rangle = \langle x,l \rangle, \forall (w,x),(k,l) \in \mathrm{A}$
Unitary if relation A is an isometry and $D(\mathrm{A}) = R(\mathrm{A}) = \mathsf{H}$

Definition 4 [12] Let A isa relation on H, then

Surjective if $R(\mathrm{A})=\mathrm{H}$

Injective if $N(\mathrm{A})=\{0\}$

Bounded if $D(\mathrm{A})=\mathrm{H}$ and $\|\mathrm{A}\|<\infty$

3. Result and Discussion

The resolvent set of A is defined in Acharya [9] to be the set $\rho(\mathrm{A})=\{z\in\mathbb{C}:(z-\mathrm{A})^{-1}$ is bounded linear operator} and its complement is the spectrum of A. The spectrum of A is denoted $\sigma(\mathrm{A})$. A scalar z such that $N(z-\mathrm{A})\neq\{0\}$ is called an eigenvalue of A. Furthermore, An non zero vector $\boldsymbol{w}$ is called an eigenvector of A related to an eigenvalue z. A set of all eigenvalues of A is said the point spectrum $\sigma_p(\mathrm{A})$.Clearly, if z is an eigenvalue of A, then z is elements of spectrum of A [1].

We give the following some Theorems of spectral theory of linear relations.

Theorem 1

Given a relation A on H is a self-adjoint. If w is an eigenvector of A corresponding to the eigenvalue z and β is an non zero scalar, then βw is also an eigenvector of A corresponding to the same eigenvalue.

Proof. Given w is an eigenvector of A corresponding to the eigenvalue z. Clearly, a relation $\mathrm{A}=\{(w,x):w,x\in\mathrm{H}\}$ give $x\in\mathrm{A}w$. Therefore, let β is an non zero scalar, then $(\beta w,z\beta w)\in\mathrm{B}$. Clearly, $(\beta w,z\beta w)=(\beta w,\beta zw)\in\beta\mathrm{A}\subset$ Consequently,we get βw is an eigenvector of A corresponding to the same eigenvalue.

Theorem2

A relation A on H is a self-adjoint. If w is an eigenvector of A then w cannot correspond to more than one eigenvalue of A.

Proof. Take any $w\neq 0$. Let z_1 and z_2 are two distinct eigenvalues of A corresponding to an eigenvector w. Furthermore, we have $(w,z_1w)\in\mathrm{A}_1$ and $(w,z_2w)\in\mathrm{A}_2$. So that, we get $\mathrm{A}_1-\mathrm{A}_2=\{(w,z_1w-z_2w):(w,z_1w)\in\mathrm{A}_1,(w,z_2w)\in\mathrm{A}_2\}$. We get $z_1w-z_2w=(z_1-z_2)w\in(\mathrm{A}_1-\mathrm{A}_2)w$. Clearly, $w\neq 0$ and $(z_1-z_2)w=0$, so that we have $z_1=z_2$.

Corrolary3A relation A on H is a self-adjoint. Eigenvectors w_1 and w_2 belonging to the two different eigenvalues of z_1 and z_2 a SALR are orthogonal.

Theorem 4 Let $\mathrm{A}\in LR(\mathrm{H})$ is a SALR that have an eigenvalue z, then an eigenspaces A of are pairwise orthogonal.

Proof.Let E_1 and E_2 is an eigenspaces of the SALR A on H corresponding to the distinct eigenvalues A_1 and A_2. Let $w_1\in E_1$ and $w_2\in E_2$ so that $\mathrm{A}w_1=zw_1$ and $\mathrm{A}w_2=zw_2$. Furthermore, we get

$$z_1\langle w_1,w_2\rangle=\langle z_1w_1,w_2\rangle=\langle \mathrm{A}w_1,w_2\rangle=\langle w_1,\mathrm{A}^*w_2\rangle \tag{4}$$

and

$$\mathrm{A}^*w_2=\overline{z_2}w_2. \tag{5}$$

From (4) and (5), we have

$$z_1\langle w_1,w_2\rangle=\langle w_1,\overline{z_2}w_2\rangle\Leftrightarrow z_1\langle w_1,w_2\rangle=z_2\langle w_1,w_2\rangle \tag{6}$$

$$\Leftrightarrow(z_1-z_2)\langle w_1,w_2\rangle=0.$$

Clearly, $z_1\neq z_2$ so that $\langle w_1,w_2\rangle=0$. Hence $w_1\perp w_2$ for each $w_1\in E_1$ and $w_2\in E_2$. Thus $E_1\perp E_2$.

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A SALR are widely applied to quantum theory, such as the determination of eigenvalues and eigenvectros on homogenous Abstract Cauchy problems. We give the following examples of homogenous Cauchy problem where M is not invertible.

Example5:

Consider the Homogenous abstract Cauchy problem for a linear system on a Hilbert space of continuous function $C[0,1]$:

$$\begin{aligned} &\dot{r}_1(t) = 2r_1(t) - r_2(t) + r_3(t) \\ &\dot{r}_2(t) = r_1(t) + r_3(t) \\ &\dot{r}_3(t) = r_1(t) - r_2(t) + 2r_3(t) \\ &r(0) = r_0 \end{aligned} \quad (7)$$

The problem (7) can be written in the form

$$\begin{aligned} &\frac{d}{dt}Mr(t) = Lr(t), t \in \mathbb{R}_+ = [0,+\infty) \\ &r(0) = r_0 \end{aligned} \quad (8)$$

where $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $L = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$. Clearly, M is invertible so that a Cauchy problem is a nondegenerate. Furthermore, we get

$$\mathrm{A} = M^{-1}L = \left\{ \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \in \mathbb{C}^3 \times \mathbb{C}^3 : x_1 = 2w_1 - w_2 + w_3, x_2 = w_1 + w_3, x_3 = w_1 - w_2 + 2w_3 \right\} \quad (9)$$

and

$$z - \mathrm{A} = \left\{ \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, z\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) : w_1, w_2, w_3 \in \mathbb{C} \right\}. \quad (10)$$

Therefore, we get $(z - \mathrm{A})w = 0 \Leftrightarrow \begin{pmatrix} z-2 & 1 & -1 \\ -1 & z & -1 \\ -1 & 1 & z-2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$. Clearly,we get $z = 1$ with $w = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $z = 2$ with $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consequently, the point spectrum of A is $\sigma_p(\mathrm{A}) = \{1,2\}$ .Thus, eigenvalues of relation A is $z = 1$ and $z = 2$, while eigenvectors of relation A is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Suppose $z \in \mathbb{C}$, we get

$$\left\| z\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\| \geq |z| \left\| \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\| - \left\| \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\| \quad (11)$$

$$\geq (|z|-2)\left\|\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right\|.$$

We can choose $C(z)=|z|-2>0 \Leftrightarrow |z|>2$. Consequently, the resolvent set of A is the set $\rho(\mathrm{A})=\{z\in\mathbb{C}:|z|>2\}$ and spectrum of A is $\sigma(A)=\{z\in\mathbb{C}:|z|\leq 2\}$.

We give the following examples of homogenous Cauchy problem where operator M is not invertible.

Example6:

Consider the homogenous abstract Cauchy problem on a Hilbert space of continuous function $C[0,1]$:

$$\frac{d}{dt}Mr(t)=Lr(t), t\in\mathbb{R}_+=[0,+\infty) \quad (12)$$

$$r(0)=r_0$$

where $M=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $L=\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$.

Clearly, M is not invertible so that Cauchy problem is a degenerate so that we get

$$\mathrm{A}=M^{-1}L=\left\{\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right): w_2=-\frac{3}{2}w_1, x_1=-w_1, \forall \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\in D(L)\right\} \quad (13)$$

and

$$z-\mathrm{A}=\left\{\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, z\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}-\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right): w_2=-\frac{3}{2}w_1, x_1=-w_1, \forall \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\in D(L)\right\}. \quad (14)$$

Therefore, we get $(z-\mathrm{A})w=0 \Leftrightarrow \begin{pmatrix} (z+1)w_1 \\ -\frac{3}{2}zw_1-x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Clearly, we get $z=-1$ and $w=\begin{pmatrix} w_1 \\ -\frac{3}{2}w_1 \end{pmatrix}$ where $w_1\neq 0$. Consequently, a point spectrum of A is $\sigma_p(\mathrm{A})=\{-1\}$. Thus, eigenvalues of relation A is $z=-1$, and eigenvectors of relation A is $\left\{\begin{pmatrix} w_1 \\ -\frac{3}{2}w_1 \end{pmatrix}: w_1\neq 0\right\}$.

Let $z\in\mathbb{C}$, we get

$$\left\|\begin{pmatrix} (z+1)w_1 \\ -\frac{3}{2}zw_1-x_2 \end{pmatrix}\right\|\geq |z|\left\|\begin{pmatrix} w_1 \\ -\frac{3}{2}w_1 \end{pmatrix}\right\|-1\left\|\begin{pmatrix} -w_1 \\ x_2 \end{pmatrix}\right\|. \quad (15)$$

$$\leq (|z|-1)\left\|\begin{pmatrix} w_1 \\ -\frac{3}{2}w_1 \end{pmatrix}\right\|.$$

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We can notfind $C(z) > 0$ such that $\left\| \begin{pmatrix} (z+1)w_1 \\ -\frac{3}{2}zw_1 - x_2 \end{pmatrix} \right\| \geq (|z|-1) \left\| \begin{pmatrix} w_1 \\ -\frac{3}{2}w_1 \end{pmatrix} \right\|$.Consequently, the resolvent set of A is the set $\rho(\mathrm{A}) = \varnothing$ and spectrum of A is $\sigma(A) = \{z : z \in \mathbb{C}\}$. A point spectrum of A is $\sigma_p(\mathrm{A}) = \{-1\} \subset \sigma(\mathrm{A})$.

4. Conclusion

An eigenvalues and eigenvectors of abstract Cauchy problems can be determined by linear relations. A relation A on H is a self-adjoint. The following are the properties of spectral theory of SALRon H. If $w$is an eigenvector of A corresponding to the eigenvalue z and β is an non zero scalar, then βw is also an eigenvector of A corresponding to the same eigenvalue. If $w$is an eigenvector of A then w cannot correspond to more than one eigenvalue of A. Consequently, eigenvectors w_1 and w_2 belonging to the two different eigenvalues of z_1 and z_2 a SALR are orthogonal.Let $\mathrm{A} \in LR(\mathrm{H})$ is a SALR that have an eigenvalue z, then an eigenspaces A of are pairwise orthogonal.

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