Additional conditions of self-adjoint operator to be applied self-adjoint linear relation on a Hilbert space

by Susilo Hariyanto

Additional conditions of self-adjoint operator to be applied self-adjoint linear relation on a Hilbert space

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Abstract. Let $\mathfrak{H}$ is a Hilbert space over field real number $\mathbb{R}$. An operator T on $\mathfrak{H}$ is a function from $\mathfrak{H}$ to $\mathfrak{H}$. A Self-adjoint operator is an operator that satisfies $T = T^*$. Furthermore, a linear relation $\mathcal{E}$ on $\mathfrak{H}$ is the set of pairs of elements w and x with $w, x \in \mathfrak{H}$. A self-adjoint linear relation is a relation that meets $\mathcal{E} = \mathcal{E}^*$. Some properties of a Self-adjoint operator on $\mathfrak{H}$ is not applicable in self-adjoint linear relation. This paper aims to determine the properties of a self-adjoint linear relation based on linear operators.

1. Introduction

Arens [1] was the first to introduce linear relation. Recently, there are many studies related to linear relation as reported by Acharya [2], Hassi et al [3], Baskakov and Chernyschov [4], Kascic [5], Gheorghe and Vasilescu [6], and Popovici and Sebestyen [7]. Arens [1] developed self-adjoint linear relation (SALR) based on a self-adjont operator (SAO). Acharya [2] developed some properties of symmetric linear relation which was self-adjoint based on Cayley transformation. Hassi, et al [3] found that a linear relation can be seen as addition of closable operators and singular relation where the closure is a Cartesian product of closed subspace. Hassi, et al [3] found the canonical decomposition properties of a linear relation based linear operator. Kascic [5] found an error in the properties of the linear relation in Arens paper. Furthermore, Kascic rebuilds in a weakened form and applies in a closed linear relation polynomial. Sandovici and Sebestyen [8] found the factorization properties of linear relation in linear space and condition that resulted in the similarity of two linear relations. Miranda [9] found the closure properties of addition and product of linear relations. Langer and Textorious [10] developed SALR by Cayley transformation and Q-function. Sari, et al [12] generalized linear relation as bounded linear operator. The application of linear relation in determining solutions, eigenvalues, and eigenvectors in differential equations can be seen in Baskakov and Chernyschov [4], Gheorghe and Vasilescu [6], and Cross [13].

A linear relation $\mathcal{E}$, or relation for short, on $\mathfrak{H}$ is defined by $\mathcal{E} = \{(w, x), w, x \in \mathfrak{H}\}$ that is a subspace of $\mathfrak{H} \times \mathfrak{H}$. A relation is named self-adjoint if $\mathcal{E} = \mathcal{E}^*$. An adjoint relation $\mathcal{E}^*$ is given by $\mathcal{E}^* = \{(r, s) \in \mathfrak{H}^2 : \langle x, r \rangle = \langle w, s \rangle, \forall (w, x) \in \mathcal{E}\}$. The graph of operator is an example of a linear relation.

A relation has been widely applied to the problem of determining eigenvalues of a cauchy problem in quantum theory. An Cauchy problems are often found in biology, physics, chemistry, finance, engineering, environment, industry, ecology and others. Given a homogenous abstract Cauchy problem on $\mathfrak{H}$

$$\frac{d}{dt} M r(t) = L r(t), t \in \mathbb{R}_+ = [0, +\infty) \tag{1}$$
$$r(0) = r_0$$

where M and L are linear operator on $\mathfrak{H}$. If $N(M) = \{0\}$ then the problem (1) is named non-degenerate. If $N(M) \neq \{0\}$ then the problem (1) is named degenerate. The (1) Cauchy problem can be made in the form

$$\frac{d}{dt} r(t) = M^{-1} L r(t), t \in \mathbb{R}_+ = [0, +\infty) \tag{2}$$
$$r(0) = r_0$$

and M can be seen as an operator on $\mathfrak{H}$. If $N(M) \neq \{0\}$ then M can not be seen as an eigenvalue operator, so that the problem (2) take form

$$\frac{d}{dt} r(t) \in M^{-1} L r(t), t \in \mathbb{R}_+ = [0, +\infty) \tag{3}$$
$$r(0) = r_0$$

where $M^{-1} L$ is a relation on $\mathfrak{H}$. The properties of SAO are not all applicable in SALR. This paper aims to give an additional condition for some properties of SAO that can not be applied in SALR to apply in SALR and determine some properties of SAO that can be applied in SALR. This paper consists of two sections as follows. We give concept and notation of a relation used on $\mathfrak{H}$ in section 2. We give some properties of SALR on $\mathfrak{H}$ based on SAO in section 3

2. Preliminaries
In this section given some properties of SAO on $\mathfrak{H}$ can be seen in [16] and notations of relation on $\mathfrak{H}$ as can be seen in [1,2,3,11].

2.1. Self-Adjoint Operator
A Hilbert space $\mathfrak{H}$ in this paper is assumed over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). An operator on $\mathfrak{H}$ is function from $\mathfrak{H}$ to $\mathfrak{H}$. An operator on $\mathfrak{H}$ is named linear if $L(w + x) = L(w) + L(x)$ and $L(\alpha w) = \alpha L(w)$ for all $w, x \in D(L)$ and scalars $\alpha \in \mathbb{K}$. If there exists a real number C such that $\|Lw\| \leq C \|w\|$ then an operator on $\mathfrak{H}$ is named bounded if there is L on $\mathfrak{H}$ is named self-adjoint if $L^* = L$.

The following lemma and theorems of SAO, together with their proof can be found [16].

Theorem 1 Let L and S are SAO. Then LS is self-adjoint if and only if the operators commute,
$$LS = SL. \tag{4}$$

Proposition 2 Let L be a bounded linear operator on $\mathfrak{H}$. Then $L = 0$ if and only if $\langle Lw, w \rangle = 0, \forall w \in \mathfrak{H}$.

Theorem 3 Let L be a bounded linear operator on $\mathfrak{H}$. Then $L^* L$ is positive.

Theorem 4 Let L and S be positive SAO with $0 \leq L^2 \leq S^2$. Then $0 \leq L \leq S$ is positive.

2.2. Linear Relation

A linear relation, or relation for short, on $\mathfrak{H}$ is denoted $\mathcal{E} = \{(w,x), w,x \in \mathfrak{H}\}$, that is a subspace of $\mathfrak{H} \oplus \mathfrak{H}$. The class of all linear relation on $\mathfrak{H}$ will be denoted by $LR(\mathfrak{H})$.

The domain of $\mathcal{E}$ is defined by $D(\mathcal{E}) = \{w \in \mathfrak{H}, (w,x) \in \mathcal{E}\}$. The range of $\mathcal{E}$ is defined by $R(\mathcal{E}) = \{x \in \mathfrak{H}, (w,x) \in \mathcal{E}\}$. The kernel of $\mathcal{E}$ is defined by $N(\mathcal{E}) = \{w \in \mathfrak{H}, (w,0) \in \mathcal{E}\}$. The multivalued part of $\mathcal{E}$ is defined by $M(\mathcal{E}) = \{x \in \mathfrak{H}, (0,x) \in \mathcal{E}\}$. The inverse of $\mathcal{E}$ is a relation of $\mathcal{E}^{-1}$ denoted by $\mathcal{E}^{-1} = \{(x,w), (w,x) \in \mathcal{E}\}$. Furthermore, the duality of $\mathcal{E}$ and its inverse $\mathcal{E}^{-1}$ is given by

$$D(\mathcal{E}^{-1}) = R(\mathcal{E}), R(\mathcal{E}^{-1}) = D(\mathcal{E}), N(\mathcal{E}^{-1}) = M(\mathcal{E}), M(\mathcal{E}^{-1}) = N(\mathcal{E}). \qquad (5)$$

An adjoint relation $\mathcal{E}^*$ is a closed relation given by $\mathcal{E}^* = \{(r,s) \in \mathfrak{H}^2 : \langle x,r \rangle = \langle w,s \rangle, \forall(w,x) \in \mathcal{E}\}$. Let $\mathcal{E}, \mathcal{J} \in LR(\mathfrak{H})$, then the sum $\mathcal{E} + \mathcal{J}$ is a relation on $\mathfrak{H}$ defined by

$$\mathcal{E} + \mathcal{J} = \{(w, x+l) : (w,x) \in \mathcal{E}, (w,l) \in \mathcal{J}\}. \qquad (6)$$

The product (composition) $\mathcal{J}\mathcal{E}$ is a relation on $\mathfrak{H}$ defined by

$$\mathcal{J}\mathcal{E} = \{(w,l) : \exists x \in \mathfrak{H}, (w,x) \in \mathcal{E}, (x,l) \in \mathcal{J}\}. \qquad (7)$$

The relation $z\mathcal{E}$ for $z \in \mathbb{K}$ is defined by $z\mathcal{E} = \{(w, zx) : (w,x) \in \mathcal{E}\}$ and the relation $z - \mathcal{E}$ is defined by $z - \mathcal{E} = \{(w, zw - x) : (w,x) \in \mathcal{E}\}$.

A relation $\mathcal{E}$ is named symmetric if $\mathcal{E} \subset \mathcal{E}^*$. A relation $\mathcal{E}$ is named self-adjoint if $\mathcal{E} = \mathcal{E}^*$. If $\langle w,r \rangle = \langle x,s \rangle, \forall(w,x), (r,s) \in \mathcal{E}$, then a relation $\mathcal{E}$ is an isometry. Furthermore, If relation $\mathcal{E}$ is an isometry and $D(\mathcal{E}) = R(\mathcal{E}) = \mathfrak{H}$ then $\mathcal{A}$ is unitary.

A relation $\mathcal{E}$ is a graph of an operator if only if $M(\mathcal{E}) = \{0\}$. If $R(\mathcal{E}) = \mathfrak{H}$, then a relation $\mathcal{E}$ is a surjective. If $N(\mathcal{E}) = \{0\}$, then a relation $\mathcal{E}$ is an injective. If $D(\mathcal{E}) = \mathfrak{H}$ and $\|\mathcal{E}\| < \infty$ then $\mathcal{E}$ is bounded.

3. Result and Discussion

In this section we give an additional condition for some properties of SAO that can not applied in SALR to apply in SALR and determine some properties of SAO that can be applied in SALR. Theorem 1 and Proposition 2 are not applicable in SALR. We give the following example Theorem 1 is not applicable in SALR.

Consider the example shows that the properties $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$ is not applicable in relations.

Example 5 Given SALR $\mathcal{E} = \{(1,3),(2,4),(3,5),(4,6),(5,7)\}$ and $\mathcal{J} = \{(1,1),(2,4),(3,9),(4,16),(5,25)\}$. Clearly, $\mathcal{E}\mathcal{J} = \{(1,3),(2,6)\}$ and $\mathcal{J}\mathcal{E} = \{(1,9),(2,16),(3,25)\}$.

Clearly, $\mathcal{E}\mathcal{J} \neq \mathcal{J}\mathcal{E}$. Given SALR $\mathcal{A}, \mathcal{B} \in LR(\mathfrak{H})$, where $\mathcal{A} = \left\{ \left(w, \frac{1}{2}w\right), w \in \mathbb{R} \right\}$ and $\mathcal{B} = \left\{ \left(\frac{1}{2}w, w^2\right), w \in \mathbb{R} \right\}$. Furthermore, $\mathcal{B}\mathcal{A} = \left\{ (w, w^2), w \in \mathbb{R} \right\}$ but $\mathcal{A}\mathcal{B} = \left\{ \left(\frac{1}{2}w_B, \frac{1}{2}w_A\right) : w_A = w_B^2 \right\}$.

Clearly, $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$. Given SALR $\mathcal{J}, \mathcal{K} \in LR(\mathfrak{H})$, where $\mathcal{J} = \{(1,0),(2,1),(3,2),(4,3)\}$ and $\mathcal{J} = \mathcal{K}$. Furthermore, $\mathcal{J}\mathcal{K} = \{(2,0),(3,1),(4,2)\}$ but $\mathcal{K}\mathcal{J} = \{\{(2,0),(3,1),(4,2)\}\}$. Clearly, $\mathcal{J}\mathcal{K} = \mathcal{K}\mathcal{J}$.

Consequently, Theorem 1 can not be applied in SALR.

Based on example 5, we give additional condition in Theorem 1 to apply in SALR. We give the following theorem.

Theorem 6 Let relation $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ are a self-adjoint. The product of two bounded SALR $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ is self-adjoint, $\mathcal{J} \subset \mathcal{E}, D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$, if and only if $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$.

Proof. $(\Rightarrow)$ Let $(w,x) \in \mathcal{E}$, so that $w \in D(\mathcal{E}) = D(\mathcal{J})$. Then there exists $x_1 \in R(\mathcal{J})$ such that $(w,x_1) \in \mathcal{J} \subset \mathcal{E}$. Furthermore, $(0, x_1 - x) = (w, x_1) - (w, x) \in \mathcal{E}$, so that $x_1 - x \in M(\mathcal{E}) = M(\mathcal{J})$. Clearly, we get $(0, x_1 - x) \in \mathcal{J}$. Therefore, we get $(w,x) = (w,x_1) - (0, x_1 - x) \in \mathcal{J}$. Consequently, since $\mathcal{E} = \mathcal{J}$, obtained $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$. Thus, the product of two bounded SALR $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ is self-adjoint, $\mathcal{J} \subset \mathcal{E}, D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$, then $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$.

$(\Leftarrow)$ Let relation $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ are a self-adjoint. Clearly, we have $\mathcal{E}\mathcal{J} = \{(k,x) : \exists w = l, \forall (w,x) \in \mathcal{E}, (k,l) \in \mathcal{J}\}$ and $\mathcal{J}A = \{(w,l) : \exists k = x, \forall (w,x) \in \mathcal{E}, (k,l) \in \mathcal{J}\}$. Since $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$, we have $k = w$ and $l = x$. Consequently, $\mathcal{J} \subset \mathcal{E}$, $D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$. Furthermore, $(k,x) \in \mathcal{E}\mathcal{J}$ so that $(k,x) \in (\mathcal{E}\mathcal{J})^*$. Since $\mathcal{E}\mathcal{J} = (\mathcal{E}\mathcal{J})^*$, we have a relation $\mathcal{E}\mathcal{J}$ is self-adjoint. Furthermore, $(w,l) \in \mathcal{J}\mathcal{E}$ so that $(w,l) \in (\mathcal{J}\mathcal{E})^*$. Since $\mathcal{J}\mathcal{E} = (\mathcal{J}\mathcal{E})^*$, we have a relation $\mathcal{J}\mathcal{E}$ is self-adjoint. Thus, if $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$ then the product of two bounded SALR $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ is self-adjoint, $\mathcal{J} \subset \mathcal{E}, D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$.

Thus, the product of two bounded SALR $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ is self-adjoint, $\mathcal{J} \subset \mathcal{E}, D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$, if and only if $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$.

We give the following example that Proposition 2 is not applicable in SALR.

Example 7:

Given relation $\mathcal{E}_1 = \{(w,0) : w \in \mathbb{R}\}$. A relation $\mathcal{E}$ on $\mathfrak{H}$ is self-adjoint, that is $\mathcal{E}_1 = \mathcal{E}_1^*$. Furthermore, $0 \in \mathcal{E}_1 w$ then $\langle \mathcal{E}_1 w, w \rangle = 0, \forall w \in \mathbb{R}$. Otherwise, given relation $\mathcal{E}_2 = \{(4,-2),(4,2),(0,0),(1,-1),(1,1)\}$. Furthermore, we get $\langle \mathcal{E}_2 w, w \rangle = 4.1 + 4(-2) + 0.0 + 1.(-1) + 1.1 = 0$. Clearly, there exists $\mathcal{E}_2 w \neq 0$ so that $\langle \mathcal{E}_2 w, w \rangle = 0$.

Based on example 7, we give condition in Proposition 2 to be applied in SALR. We give the following theorem.

Proposition 8 Let SALR $\mathcal{E}$ is a bounded on $\mathfrak{H}$. If $0 \in \mathcal{E}w$ then $\langle \mathcal{E}w, w \rangle = 0, \forall w \in \mathfrak{H}$.

Proof. A relation $\mathcal{E}$ is self-adjoint that is $\mathcal{E} = \mathcal{E}^*$. Furthermore, if $0 \in \mathcal{E}w$ then $\langle \mathcal{E}w, w \rangle = \mathcal{E}w_1.w_1 + \mathcal{E}w_2.w_1 + \ldots + \mathcal{E}w_n.w_n = 0$. Thus, if $0 \in \mathcal{E}w$ then $\langle \mathcal{E}w, w \rangle = 0, \forall w \in \mathfrak{H}$.

Let a relation $\mathcal{E}$ is self-adjoint. We give that $\mathcal{E}$ is positive denoted $\mathcal{E} \geq 0$ if $\langle x, w \rangle \geq 0$ for each $w, x \in \mathfrak{H}$. We give the following properties of SALR.

Theorem 9 Let relation $\mathcal{E}$ on $\mathfrak{H}$ is self-adjoint. The following statements are true.

1. $\|\mathcal{E}\| \leq 1$ if and only if $-I \leq \|\mathcal{E}\| \leq I$.

2. $O \leq \|\mathcal{E}\| \leq I \Leftrightarrow O \leq \|\mathcal{E}\|$ and $\|\mathcal{E}\| \leq 1 \Leftrightarrow \mathcal{E}^2 \leq \mathcal{E}$.

Proof. (1)Let relation $\mathcal{E}$ is self-adjoint then $\langle x, w \rangle \in \mathbb{R}$ get

$$\langle x - w, w \rangle = \langle x, w \rangle - \|w\|^2 \tag{8}$$
$$\leq |\langle x, w \rangle| - \|w\|^2$$
$$\leq (\|x\| - \|w\|)\|w\|$$

If $\|x\| \le \|w\|$ for each $x, w \in \mathfrak{H}$ then $(\mathcal{E} - I) \le O$ which is $\|\mathcal{E}\| \le 1$ implies $-I \le \|\mathcal{E}\| \le I$. Otherwise, If $(\mathcal{E} - I) \le O$ then $\langle x, w \rangle \le \|w\|^2$. Furthermore, we have $\sup_{\|w\|=1} |\langle x, w \rangle| \le 1$. Since $\mathcal{E}$ is self-adjoint then $\|\mathcal{E}\| = \sup_{\|w\|=1} |\langle x, w \rangle|$. Furthermore, $-I \le \|\mathcal{E}\| \le I$ implies $\|\mathcal{E}\| \le 1$.

(2) Let $O \le \|\mathcal{E}\| \le I$ if and only if $O \le \|\mathcal{E}\|$ and $\|\mathcal{E}\| \le 1$. If $O \le \|\mathcal{E}\|$ and $\|\mathcal{E}\| \le 1$ then $\mathcal{E}$ is a nonnegative contractive. Clearly, $\langle x, x \rangle = \|x\|^2 \le \langle x, w \rangle$ for some $w, x \in \mathfrak{H}$, so that $\mathcal{E}^2 \le \mathcal{E}$. Conversely, if relation $\mathcal{E}$ on $\mathfrak{H}$ is self-adjoin and $\mathcal{E}^2 \le \mathcal{E}$ then $\|x\|^2 = \langle x, x \rangle \le \langle x, w \rangle \le \|x\| \|w\|$ and hence $\|x\| \le \|w\|$ that is $\|\mathcal{E}\| \le 1$. Thus $O \le \|\mathcal{E}\|$ and $\|\mathcal{E}\| \le 1$ if and only if $\mathcal{E}^2 \le \mathcal{E}$.

Theorem 10 Let $\mathcal{J}$ and $\mathcal{E}$ be positive SALR on $\mathfrak{H}$ with $0 \le \mathcal{E}^2 \le \mathcal{J}^2$. Then $0 \le \mathcal{E} \le \mathcal{J}$ is a positive.

Proof. Let $\mathcal{J} = \{(w, x) : w, x \in \mathfrak{H}\}$ and $\mathcal{E} = \{(r, s) : r, s \in \mathfrak{H}\}$ be positive SALR with $0 \le \mathcal{A}^2 \le \mathcal{B}^2$. Clearly, we get

$$0 \le \mathcal{E}^2 \le \mathcal{J}^2 \Leftrightarrow 0 \le \langle s, r \rangle^2 \le \langle x, w \rangle^2. \tag{9}$$
$$\Leftrightarrow 0 \le \langle s, r \rangle \le \langle x, w \rangle$$
$$\Leftrightarrow 0 \le \mathcal{E} \le \mathcal{J}.$$

Furthermore, if $\langle s, r \rangle^2$ is a positive then $\langle s, r \rangle$ is also positive, if $\langle s, r \rangle^2$ is a positive then $\langle s, r \rangle$ is also positive. Thus, $0 \le \mathcal{E} \le \mathcal{J}$ is a positive.

Theorem 11 Let a relation $\mathcal{E}$ is self-adjoint on $\mathfrak{H}$. Then $\mathcal{E}^* \mathcal{E}$ is positive.

Proof. Given $(w, x) \in \mathcal{E}$ and $(k, l) \in \mathcal{E}^*$. A relation $\mathcal{E}$ on $\mathfrak{H}$ is self-adjoint, that is $\mathcal{E} = \mathcal{E}^*$. Clearly, $\mathcal{E}^* \mathcal{E} = \{(w, l) : x = k, (w, x) \in \mathcal{E}, (k, l) \in \mathcal{E}^*\}$ implies $\langle (\mathcal{E}^* \mathcal{E}) w, w \rangle = \langle \mathcal{E}^* w, \mathcal{E}^* w \rangle = \langle l, l \rangle = \|l\|^2 \ge 0$. Thus, $\langle (\mathcal{E}^* \mathcal{E}) w, w \rangle \ge 0$ then $\mathcal{E}^* \mathcal{E}$ is positive. Thus, let a relation $\mathcal{E}$ is self-adjoint on $\mathfrak{H}$, then $\mathcal{E}^* \mathcal{E}$ is positive.

Conclusion

There are properties of linear operators that cannot be applied in SALR. Additional requirements are needed to apply to relations, so that we get the properties of relations. Let relations $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ are a self-adjoint. The product of two bounded self-adjoint relations $\mathcal{E}$ and $\mathcal{J}$ on $\mathfrak{H}$ is self-adjoint, $\mathcal{J} \subset \mathcal{E}, D(\mathcal{E}) = D(\mathcal{J})$, and $M(\mathcal{E}) = M(\mathcal{J})$, if and only if $\mathcal{E}\mathcal{J} = \mathcal{J}\mathcal{E}$. Let SALR $\mathcal{E}$ is a bounded on $\mathfrak{H}$. If $0 \in \mathcal{E}w$ then $\langle \mathcal{E}w, w \rangle = 0, \forall w \in \mathfrak{H}$. There are properties of linear operators that can be applied in linear relations, including Theorem 9, Theorem 10, and Theorem 11.

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