A convergence theorem on the dunford integral

by Solikhin Solikhin

Submission date: 13-Apr-2023 11:47AM (UTC+0700)
Submission ID: 2063208939
File name: khin_A_Convergence_Theorem_on_the_Dunford_Integral_organized.pdf (554.13K)
Word count: 3309
Character count: 14288

A convergence theorem on the dunford integral

Solikhin, A Aziz, YD Sumanto, R Heri SU

Department of Mathematics, Faculty of Science and Mathematics Diponegoro University, Jl. Prof.Soedarto, S.H. Semarang, Indonesia 50275

Corresponding author: solikhin@live.undip.ac.id

Abstract. This article discusses the convergence theorem of the Dunford integrals. We examine the sufficient conditions so that limit of the sequence of integral value whose Dunford integrable is same as limit of functions sequence. We have obtained that to guarantee a function to be Dunford integrable and its limit of functions sequence are same as value of the functions, then a sequence of Dunford integrable function is uniform convergent or weakly convergent, weakly monoton, and its limit exist. Furthermore, its weakly convergent and bounded.

1. Introduction

The Measurable function f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable. It is an absolute integral while Henstock-Kurzweil (Henstock) integral is not. Its means that there is a function g that is Henstock integrable and its absolute value is not Henstock integrable. In the late 1960s, McShane introduces an integral known as the McShane integral. The same as the Lebesgue integral, The McShane integral is an absolute integral. They are equivalent [1]. Some of the Lebesgue integral applications are differentiation and integration, mathematical models for probability, and convergence and limit theorems [2]. While, applications of the Henstock integral are in ordinary differential equations [3] , in financial market modelling [4], and others.

The Lebesgue integral is used to define the Dunford integral. The Dunford integral is defined over a weakly measurable function which the function is Banach-valued function such that for each x^* belong to X^* the function $x^*(f)$ is Lebesgue integrable [5]. Some properties of Dunford integral has been discussed by [5]. A collection of all Dunford integrable functions is linear space and seminorm space[5]. Furthermore, operators which work on space of the Dunford integrable function is linear and bounded operators [5]. It is weakly compact linear operators[5], [6].

Let function $f_1, f_2, f_3, ...$ are integrable on $[a,b]$, and suppose $f_1, f_2, f_3, ...$ converges pointwise to function f . Is f integrable on $[a,b]$ and does $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$? Any theorem that provides conditions for this question to have an affirmative answer is known as a convergence theorem. In Lebesgue integral, the best possible theorem is the dominated convergence theorem [1], [7]. In other hand, there are the uniform convergence theorem, the bounded convergence theorem, the monotone convergence theorem, and Fatou's lemma. While on the Henstock integral, the controlled convergence theorem is the best theorem [7] - [9]. The hypotheses of this theorem will then be extended to establish convergence theorems for the Dunford integrals. Let $\{f_1, f_2, f_3, ...\}$ be a sequence of Dunford integrable function.

We examine the sufficient conditions so that limit of the sequence of integral value whose Dunford integrable is same as limit of functions sequence.

2. Dunford integral

We shall now define the concept of sequence of convergence function and weakly measurable function, which will be basic in Dunford integral and some convergence theorems.

Definition 2.1 *Let $f_1, f_2, f_3, \ldots,$ be a function from $[a,b]$ to X. A sequence of function $\{f_n\}$ is convergent to function f if $\{f_n(x)\}$ is convergent to $f(x)$, i.e. there is an $f(x)$ element X and for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ such that if $n \geq n_0$ implies*

$$\| f_n(x) - f(x) \|_X < \varepsilon .$$

So, $\{f_n\}$ is convergent in $x \in [a,b]$ means $\{f_n(x)\}$ is convergent.

If a sequence of function $\{f_n\}$ is convergent at each x belong to $[a,b]$, then $\{f_n\}$ is convergent on $[a,b]$.

Definition 2.2 *A sequence of function $\{f_n\}$ is said to be weakly convergent to function f at $x \in [a,b]$, if $\{f_n(x)\}$ is weakly convergent to $f(x) \in X$, i.e. for every $\varepsilon > 0$ and $x^* \in X^*$ there is an $n_0 = n_0(\varepsilon, x^*, x) \in \mathbb{N}$ such that if $n \geq n_0$ implies*

$$| x^* f_n(x) - x^* f(x) | < \varepsilon .$$

And $\{f_n\}$ is said to be weakly convergent to function f on $[a,b]$, if for every $x \in [a,b]$ and $\varepsilon > 0$, $x^* \in X^*$ there is an $n_0 = n_0(\varepsilon, x^*, x) \in \mathbb{N}$ such that if $n \geq n_0$ implies

$$| x^* f_n(x) - x^* f(x) | < \varepsilon .$$

A sequence of function $\{f_n\}$ is said to be weakly uniform convergent to f, if for every $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $n \geq n_0$ implies

$$| x^* f_n(x) - x^* f(x) | < \varepsilon$$

for every $x \in [a,b]$ and $x^* \in X^*$.

Definition 2.3 *A sequence of function $\{f_n\}$ is called weakly increasing monotone on $[a,b]$, if for every x^* element X^*, $\{x^*(f_n)\}$ is increasing monotone, i.e $x^* f_i(x) \leq x^* f_{i+1}(x)$ for all $i \in \mathbb{N}$ and for every $x \in [a,b]$, $x^* \in X^*$. A sequence of function $\{f_n\}$ is called weakly descreasing monotone on $[a,b]$, for every x^* element X^*, $\{x^*(f_n)\}$ is descreasing monotone, i.e $x^* f_i(x) \geq x^* f_{i+1}(x)$ for all $i \in \mathbb{N}$ and for every x belong to $[a,b]$, $x^* \in X^*$.*

A sequence of weakly increasing monotone or weakly descreasing monotone are called a sequence of weakly monotone.

Measurable function can be defined by simple functions. As follows.

Definition 2.4 *A function $f : [a,b] \to X$ is called measurable, if there is a simple sequence of functions $\{f_k\}, k \in \mathbb{N}$* such that

$$\lim_{k\to\infty}\left\|f_k(x)-f(x)\right\|_X=0,$$

for every x belong to $[a,b]$.

Definition 2.5 *A functions* f *is called weakly measurable, if for each* x^* *element* X^* *we have* $x^*(f)$ *is measurable*.

The Dunford integral can be defined on basic measurable function. It is related to Lebesgue integral.
Definition 2.6 *Let* f *is weakly measurable. If* $x^*(f):[a,b]\to R$ *is Lebesgue integrable for each* x^* *element* X^*, *then function* f *is said to be Dunford integrable.*
The Dunford integral $(D_L)\int_A f$ of f over measurable set $A\subset[a,b]$ is defined by vector $x^{**}_{(f,A)}$ element X^{**}, i.e.

$$x^{**}_{(f,A)}(x^*)=\int_A x^*(f),$$

for all x^* element X^*.
The set of all function f which it is the Dunford integrable denoted by $D_L[a,b]$.
We have shown that $D_L[a,b]$ is hold the additive and homogenous properties.

Definition 2.7 *Afunction* $f:[a,b]\to R$ *is called McShane integrable on* $[a,b]$, *if there exists real number* $M_C\in R$ *such that for every* $\varepsilon>0$ *there exists a positive function* δ *and if* $\mathcal{M}=\{(D,x)\}$ *McShane partitions* δ*-fine on* $[a,b]$ *implies*

$$\left|\sum_{x\in[a,b]} f(x)\alpha(D)-M_C\right|<\varepsilon.$$

The McShane integral and the Lebesgue integral are equivalent.. By Definition 2.6, we obtained that f is Dunford integrable iff $x^*(f)$ is Lebesgue integrable for all x^* element X^*. Therefore, we can say f is Dunford integrable iff $x^*(f)$ is McShane integrable for all x^* element X^*.

3. Convergence theorems

Now we consider the convergence theorems for Dunford integrals. Let X is a complete normed space which X^* is dual space of X and X^{**} is the seon dual space of X. Lets $[a,b]\subset R$ is closed interval, and functions $f,f_n:[a,b]\to X$ $\forall n\in\mathbb{N}$. If $f_n\in D_L[a,b],\forall n\in\mathbb{N}$ and $\{f_n\}$ is weakly convergent to f on $A\subset[a,b]$, we showed some condititions which cause $f\in D_L[a,b]$ and

$$x^{**}_{(f,A)}=\lim_{n\to\infty}x^{**}_{(f_n,A)}.$$

The following is a convergence theorems on the Dunford integrals.
Theorem 3.1 (Uniformly convergence theorem) *Lets* $f,f_n:[a,b]\to X$ *and* $f_n\in D_L[a,b]$ *for every* n*.If* $\{f_n\}$ *is weakly uniform convergent to* f *on* $A\subset[a,b]$, *then* $f\in D_L[a,b]$ *and*

$$x^{**}_{(f,A)}=\lim_{n\to\infty}x^{**}_{(f_n,A)}.$$

Proof: Let arbitrary $A \subset [a,b]$. Sequence $\{f_n\}$ is weakly uniform convergent to f on $A \subset [a,b]$. This means, for every $\varepsilon > 0$ there exists nature number $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ we have

$$\left|x^* f_n(x) - x^* f(x)\right| < \frac{\varepsilon}{5\alpha(A)+1}$$

for almost all x element $[a,b]$ and x^* element X^*.

Functions $f_n \in D_L[a,b]$, $\forall n \in \mathbb{N}$, i.e. for $\varepsilon > 0$, $x^* \in X^*$, and $A \subset [a,b]$ then $x^*(f_n)$ is Lebesgue integrable (is equivalent McShane integrable) on A. So that, there exists $\delta_n > 0$ on $[a,b]$ and if $\mathcal{D}_1 = \{(D_1, x)\}$ and $\mathcal{D}_2 = \{(D_2, x)\}$ are McShane partitions δ_n-fine on A then

$$\left|\mathcal{D}_1 \sum_{x \in A} x^* f_n(x)\alpha(D_1) - x^{**}_{(f_n,A)}(x^*)\right| < \frac{\varepsilon}{5}$$

and

$$\left|\mathcal{D}_1 \sum_{x \in A} x^* f_n(x)\alpha(D_1) - \mathcal{D}_2 \sum_{x \in A} x^* f_n(x)\alpha(D_2)\right| < \frac{\varepsilon}{5}.$$

So that, if $\mathcal{D}_1 = \{(D_1, x)\}$ and $\mathcal{D}_2 = \{(D_2, x)\}$ are McShane partitions δ_n-fine on A we obtained

$$\begin{aligned}
&\left|\mathcal{D}_1 \sum_{x \in A} x^* f(x)\alpha(D_1) - \mathcal{D}_2 \sum_{x \in A} x^* f(x)\alpha(D_2)\right| \\
&\leq \left|\mathcal{D}_1 \sum_{x \in A} x^* f(x)\alpha(D_1) - \mathcal{D}_1 \sum_{x \in A} x^* f_n(x)\alpha(D_1)\right| \\
&+ \left|\mathcal{D}_1 \sum_{x \in A} x^* f_n(x)\alpha(D) - \mathcal{D}_2 \sum_{x \in A} x^* f_n(x)\alpha(D_2)\right| \\
&+ \left|\mathcal{D}_2 \sum_{x \in A} x^* f_n(x)\alpha(D_2) - \mathcal{D}_2 \sum_{x \in A} x^* f(x)\alpha(D_2)\right| \\
&< \varepsilon.
\end{aligned}$$

Its means $x^*(f)$ is Lebesgue integrable on A and for every $A \subset [a,b]$ closed interval there exists $x^{**}_{(f,A)}$ element X^{**} so

$$x^{**}_{(f,A)}(x^*) = (L)\int_A x^* f.$$

So $f \in D_L[a,b]$.

A function $f \in D_L[a,b]$, i.e. for each $x^* \in X^*$ and $A \subset [a,b]$, then real function $x^* f$ is Lebesgue integrable (McShane integrable) on A. This means for every $\varepsilon > 0$ there exixst positive function δ' on A so that if $\mathcal{D}_3 = \{(D_3, x)\}$ is McShane partition δ'-fine on A we have

$$\left|\mathcal{D}_3 \sum_{x \in A} x^* f(x)\alpha(D_3) - x^{**}_{(f,A)}(x^*)\right| < \frac{\varepsilon}{5}.$$

Take $\delta^*(x) = \text{Min}\{\delta_n(x), \delta'(x)\}$ for almost all $x \in [a,b]$. We obtained positive functions δ^* on $[a,b]$. If $\mathcal{D}_4 = \{(D_4, x)\}$ is McShane partition δ'-fine on A then

$$\begin{aligned}\left|x^{**}_{(f_n,A)}(x^*)-x^{**}_{(f,A)}(x^*)\right| &\le \left|x^{**}_{(f_n,A)}(x^*)-\mathcal{D}_4\sum_{x\in A}x^*f_n(x)\alpha(D_4)\right| \\ &+\left|\mathcal{D}_4\sum_{x\in A}x^*f_n(x)\alpha(D_4)-\mathcal{D}_4\sum_{x\in A}x^*f(x)\alpha(D_4)\right| \\ &+\left|\mathcal{D}_4\sum_{x\in A}x^*f(x)\alpha(D_4)-x^{**}_{(f,A)}(x^*)\right| \\ &<\frac{\varepsilon}{5}+\frac{\varepsilon}{5\alpha(A)+1}\mathcal{D}^*\sum_{x\in A}\alpha(D)+\frac{\varepsilon}{5} \\ &<\varepsilon .\end{aligned}$$

In other word

$$x^{**}_{(f,A)}(x^*)=\lim_{n\to\infty}x^{**}_{(f_n,A)}(x^*),$$

for every $x^*\in X^*$. □

Theorem 3.2 (Monotone congvergence theorem) *Lets* $f, f_n:[a,b]\to X$ *and* $f_n\in D_L[a,b], \forall n\in\mathbb{N}$ *. if*

(i) $\{f_n\}$ *weakly monotone on* A*,*

(ii) $\{f_n\}$ *weakly convergent to* f *on* A*, and*

(iii) $\lim_{n\to\infty}x^{**}_{(f_n,A)}(x^*)$*,* $\forall x^*\in X^*$ *exists and finite.*

Then $f\in D_L[a,b]$ *and*

$$x^{**}_{(f,A)}=\lim_{n\to\infty}x^{**}_{(f_n,A)}$$

for every $A\subset[a,b]$ *closed interval.*

Proof: we showed for sequence of increasing monotone. Lets arbitrary $\varepsilon>0$ and closed interval $A\subset[a,b]$. $\{f_n\}$ is weakly convergent to function f on $A\subset[a,b]$, i.e. for every $\varepsilon>0$, $x^*\in X^*$ and $x\in[a,b]$, there is a nature number $m_0=m_0(\varepsilon,x^*,x)$ such that if $n\ge m_0$ we have

$$\left|x^*f_n(x)-x^*f(x)\right|<\frac{\varepsilon}{5\alpha(A)+5}.$$

Functions $f_n\in D_L[a,b]$, $\forall n\in\mathbb{N}$, i.e. for every $\varepsilon>0$, $x^*\in X^*$, and $A\subset[a,b]$ then $x^*(f_n)$ is McSahne integrable on A. So that, there exists $\delta_n>0$ on $[a,b]$ and if $\mathcal{D}=\{(D,x)\}$ is McShane partitions δ_n-fine on A we have

$$\left|\mathcal{D}\sum_{x\in A}x^*(f_n)(x)\alpha(D)-x^{**}_{(f_n,A)}(x^*)\right|=\left|\mathcal{D}\sum_{x\in A}x^*(f_n)(x)\alpha(D)-(L)\int_A x^*(f_n)\right|<\frac{\varepsilon}{3}.$$

Sequence of weakly increasing monotone $\{f_n\}$ on A. Its means for each $x^*\in X^*$ sequence $\{x^*(f_n)\}$ is increasing monotone on A. Its means $x^*f_n(x)\le x^*f_{n+1}(x)$ for every x belong to $A\subset[a,b]$. Furthermore, since $f_n\in D_L[a,b]$, for each x^* element X^* and $A\subset[a,b]$ we have $x^*(f_n)$ is Lebesgue integrable on A such that

$$x^{**}_{(f_n,A)}(x^*)=(L)\int_A x^*(f_n).$$

Therefore

$$x^{**}_{(f_n,A)}\left(x^*\right)=\left(L\right)\int_A x^*\left(f_n\right)\leq\left(L\right)\int_A x^*\left(f_{n+1}\right)=x^{**}_{(f_{n+1},A)}\left(x^*\right).$$

Its means $\left\{(L)\int_A x^* f_n\right\}$ is the sequence of increasing monoton on A. We known $\lim_{n\to\infty} x^{**}_{(f_n,A)}\left(x^*\right)$ for each $x^*\in X^*$ exists and finite, then $\left\{(L)\int_A x^* f_n\right\}$ is convergent takes to L such that

$$\lim_{n\to\infty} x^{**}_{(f_n,A)}\left(x^*\right)=\lim_{n\to\infty}\left(L\right)\int_A x^*\left(f_n\right)=L.$$

So, for $\varepsilon>0$ in above, there is a $n_0=n_0\left(\varepsilon,x^*\right)\in\mathbb{N}$ such that if $n\geq n_0$ we have

$$\left|(L)\int_A x^* f_n - L\right|<\frac{\varepsilon}{5}.$$

Takes $m^*=m^*\left(\varepsilon,x^*,x\right)=\max\left\{n_0,m_0\right\}$

We construct positive function by

$$\delta\left(x\right)=\delta_{m^*}\left(x\right) \text{ for almost all } x\in[a,b] \text{ and } x^* \text{ element } X^*.$$

Therefore, if $\mathcal{D}=\left\{(D,x)\right\}$ McShane partition $\delta-$fine on A we obtained

$$\begin{aligned}\left|\mathcal{D}\sum_{x\in A} x^*\left(f\right)\left(x\right)\alpha\left(D\right)-L\right| &\leq\left|\mathcal{D}\sum_{x\in A} x^*\left(f\right)\left(x\right)\alpha\left(D\right)-\mathcal{D}\sum_{x\in A} x^*\left(f_{m^*}\right)\left(x\right)\alpha\left(D\right)\right| \\ &\quad+\left|\mathcal{D}\sum_{x\in A} x^*\left(f_{m^*}\right)\left(x\right)\alpha\left(D\right)-\left(L\right)\int_A x^*\left(f_{m^*}\right)\right|+\left|\left(L\right)\int_A x^*\left(f_{m^*}\right)-L\right| \\ &\leq \mathcal{D}\sum_{x\in A}\left|x^*\left(f\right)\left(x\right)-x^*\left(f_{m^*}\right)\left(x\right)\right|\alpha\left(D\right)+\frac{\varepsilon}{3}+\frac{\varepsilon}{5} \\ &<\varepsilon.\end{aligned}$$

Its means $x^* f$ is McShane integrable, so its Lebesgue integrable on A and there exists $x^{**}_{(f,A)}\in X^{**}$ such that

$$x^{**}_{(f,A)}\left(x^*\right)=\left(L\right)\int_A x^* f.$$

In other word, $f\in D_L[a,b]$ and $x^{**}_{(f,A)}\left(x^*\right)=L \Leftrightarrow x^{**}_{(f,A)}\left(x^*\right)=\lim_{n\to\infty} x^{**}_{(f_n,A)}\left(x^*\right)$ for every $x^*\in X^*$. □

Theorem 3.3 (Fatou's Lemma) *Lets $f,f_n:[a,b]\to X$ and $f_n\in D_L\left[a,b\right],\forall n\in\mathbb{N}$. If for every $A\subset\left[a,b\right]$ closed interval, $\left\{f_n\right\}$ is weakly convergent on A and $x^* f_n\left(x\right)\geq 0$ almost everywhere on A for all n and $x^*\in X^*$, then $f\in D_L[a,b]$ and*

$$x^{**}_{(f,A)}\left(x^*\right)\leq\liminf_{n\to\infty} x^{**}_{(f_n,A)}\left(x^*\right).$$

Proof: Lets arbitrary closed interval $A\subset[a,b]$. For all nature number n and x^* element X^* we defined $h_n:A\subset[a,b]\to X$ by

$$x^* h_n\left(x\right)=\inf_{l\geq n}\left\{x^* f_l\left(x\right)\right\} \text{ for every } x\in A\subset[a,b].$$

We obtained $\{x^* h_n\}$ is increasing monotone on A for every $x^* \in X^*$, i.e. $\{h_n\}$ is weakly increasing monotone on A.
For all $n \in \mathbb{N}$ we obtained

$$x^* h_n(x) \leq x^* f_n(x)$$

for almost all x element A. We known $x^*(f_n)(x) \geq 0$ almost everywhere, then $x^* g_n(x) \geq 0$ almost everywhere on A.
Therefore, we have

$$\lim_{n\to\infty} (L)\int_A x^*(h_n) \leq \lim_{n\to\infty} (L)\int_A x^*(f_n) \quad \Leftrightarrow \quad \lim_{n\to\infty} x^{**}_{(h_n,A)}(x^*) \leq \lim_{n\to\infty} x^{**}_{(f_n,A)}(x^*).$$

We known $\{x^* h_n\}$ is increasing monotone and $\{x^* f_n\}$ is convergent to $x^*(f)$ for each $x^* \in X^*$, then $\{x^* h_n\}$ is convergent to $x^*(f)$ for each $x^* \in X^*$. Its means $\{h_n\}$ is weakly convergent to f. Since $\lim_{n\to\infty} (L)\int_A x^* h_n$ exists, by Monotone Convergence Theorem we obtained $f \in D_L[a,b]$ and if $A \subset [a,b]$ we have

$$x^{**}_{(f,A)}(x^*) = \lim_{n\to\infty} x^{**}_{(h_n,A)}(x^*) \leq \liminf_{n\to\infty} x^{**}_{(f_n,A)}(x^*),$$

for each $x^* \in X^*$. □

Fatou Lemma results Lebesgue Dominated Convergence Theorem. As follow.

Theorem 3.4 *Lets* $f, h, f_n : [a,b] \to X$ *and* $h, f_n \in D_L[a,b]$, $\forall n \in \mathbb{N}$. *If*

(i) $\{f_n\}$ *is weakly convergent to* f *on* $[a,b]$,

(ii) $|x^* f_n(x)| \leq x^* h(x)$ $\forall n \in N$, $x^* \in X^*$, *and* $x \in [a,b]$

we have $f \in D_L[a,b]$ *and*

$$x^{**}_{(f,A)}(x^*) = \lim_{n\to\infty} x^{**}_{(f_n,A)}(x^*)$$

for every $A \subset [a,b]$ *closed interval.*

Proof: Let $A \subset [a,b]$ arbitrary closed interval.
Sequence $\{f_n\}$ is weakly convergent to function f on $[a,b]$, i.e. for every $\varepsilon > 0$, $x \in [a,b]$, and $x^* \in X^*$ there is an $n_0 = n_0(\varepsilon, x^*, x) \in \mathbb{N}$ such that if $n \geq n_0$ we have

$$|x^*(f_n)(x) - x^*(f)(x)| < \varepsilon$$

or

$$x^*(f_n)(x) - \varepsilon < x^*(f)(x) < x^*(f_n)(x) + \varepsilon.$$

We known that

$$|x^*(f_n)(x)| \leq x^*(h)(x)$$

We obtained

$$-x^*(h)(x) \leq x^*(f_n)(x) \leq x^*(h)(x)$$

Therefore

$$x^* h(x) - x^* f_n(x) = x^*(h - f_n)(x) \geq 0.$$

Sequence $\{x^*(f_n)\}$ is convergent to function $x^*(f)$ on $[a,b]$ for every $x^* \in X^*$, then $\{x^*(h - f_n)\}$ is convergent to function $x^*(h - f)$ on $[a,b]$. Its means $\{h - f_n\}$ is weakly convergent to function

$h-f$ on $[a,b]$. In the next result, the sequence $\left\{(L)\int_A x^*(h-f_n)\right\}$ is bounded for all $n\in\mathbb{N}$ and x^* element X^*.
By Fatou's Lemma, we get $h-f\in D_L[a,b]$ and

$$x^{**}_{(h-f,A)}(x^*)\le \liminf_{n\to\infty} x^{**}_{(h-f_n,A)}(x^*).$$

We are known $-x^*h(x)\le x^*f(x)\le x^*h(x)$ for every $x\in A\subset[a,b]$, $x^*\in X^*$ and $h\in D_L[a,b]$, then $f\in D_L[a,b]$ and

$$\begin{aligned}0&\le x^{**}_{(h,A)}(x^*)-x^{**}_{(f,A)}(x^*)\\&\le \liminf_{n\to\infty} x^{**}_{(h-f_n,A)}(x^*)\\&=\liminf_{n\to\infty}\left\{(L)\int_A x^*h+(L)\int_A x^*(-f_n)\right\}\\&=(L)\int_A x^*h-\limsup_{n\to\infty}(L)\int_A x^*(f_n)\\&=x^{**}_{(h,A)}(x^*)-\limsup_{n\to\infty} x^{**}_{(f_n,A)}(x^*)\end{aligned}$$

We get

$$\limsup_{n\to\infty} x^{**}_{(f_n,A)}(x^*)\le x^{**}_{(f,A)}(x^*).$$

In other hand, by $x^*h(x)+x^*f_n(x)\ge 0$. We have $\liminf_{n\to\infty} x^{**}_{(f_n,A)}(x^*)\ge x^{**}_{(f,A)}(x^*)$.
Therefore,

$$x^{**}_{(f,A)}(x^*)=\lim_{n\to\infty} x^{**}_{(f_n,A)}(x^*). \qquad \square$$

From Lebesgue Dominated Convergen Theorems, its implies theorems in below.

Theorem 3.5 (Bounded Convergence Theorem) *Lets* $f,f_n:[a,b]\to X$ *and* $f_n\in D_L[a,b],\forall n\in\mathbb{N}$ *If for every* $A\subset[a,b]$ *closed interval,* $\{f_n\}$ *is weakly convergent to* f *on* A *and there exist real number* $M>0$ *such that* $|x^*(f_n)(x)|\le M$ $\forall n\in\mathbb{N}$, $x^*\in X^*$, *and* $x\in A$, *then* $f\in D_L[a,b]$ *and we have*

$$x^{**}_{(f,A)}=\lim_{n\to\infty} x^{**}_{(f_n,A)}.$$

Proof: Let's arbitrary closed interval $A\subset[a,b]$. We take $x^*h(x)=M>0$, $M\in R$ for every $x\in A$. We known $\{f_n\}$ is weakly convergent to f on A and $|x^*f_n(x)|\le x^*h(x)=M$ $\forall n\in\mathbb{N}$, $x^*\in X^*$, and $x\in A$, by Theorem 3.4 we get $f\in D_L[a,b]$ and

$$x^{**}_{(f,A)}=\lim_{n\to\infty} x^{**}_{(f_n,A)}. \qquad \square$$

4. Conclusion
We have obtained that to guarantee a function which is Dunford integrable and its limit of functions sequence are same as the value of the functions, then a sequence of Dunford integrable function is

uniformly convergent or weakly convergent, weakly monotone, and its limit exist. Furthermore, its weakly convergent and bounded.

Acknowledgments
This research was supported by FSM Universitas Diponegoro [Project number :1958/UN7.5.8/PP/2020].

Reference

[1] Gordon R A 1994 *The Integral of Lebesgue, Denjoy, Perron, and Henstock* USA Mathematical Society

[2] Hong D, Wang J and Gardner R 2005 *Real Analysis with an Introduction to Wavelets and Applications* New York Elsevier Academic Press

[3] Lee PY and Vyborny R 2000 *An Easy Approach after Kurzweil and Henstock* New York: Cambridge University Press

[4] Krejci P, Lamba H, Monteiro G A and Rachinskii D 2016 *Mathematica Bohemica* **141** 261-286

[5] Schwabik S and Guoju Y 2005 *Topics in Banach Space Integration* Singapore World Scientific

[6] S Solikhin, S Hariyanto, Y D Sumanto and A Aziz 2020 *Journal of Physics Conference Series* 1524:012045

[7] Lee PY 1989 *Lanzhou Lectures on Henstock Integration* Singapore World Scientific

[8] Mema E 2013 *International Mathematical Forum* **8** 913-919

[9] Lee P Y and Chew T S 1985 *Bull London Math Soc* **17** 557-564

A convergence theorem on the dunford integral

ORIGINALITY REPORT

19%	9%	18%	9%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Kolokoltsov, Vassili N., and Victor P. Maslov. "Idempotent Analysis", Idempotent Analysis and Its Applications, 1997. Publication	2%
2	www.eri.u-tokyo.ac.jp Internet Source	2%
3	George Metcalfe, Nicola Olivetti, Dov Gabbay. "Proof Theory for Fuzzy Logics", Springer Science and Business Media LLC, 2009 Publication	1%
4	Submitted to National Taichung University of Education Student Paper	1%
5	Kuhnel, A.. "Equations of motion in the theory of gravitational perturbations", Annals of Physics, 19640605 Publication	1%
6	Submitted to University of North Florida Student Paper	1%
7	Submitted to East Tennessee State University Student Paper	

		1%
8	archive.org Internet Source	1%
9	opus.lib.uts.edu.au Internet Source	1%
10	Mi-Ho Giga. "Convergence Theorems in the Theory of Integration", Nonlinear Partial Differential Equations, 2010 Publication	1%
11	SOKOL BUSH KALIAJ. "A VARIATIONAL McSHANE INTEGRAL CHARACTERISATION OF THE WEAK RADON–NIKODYM PROPERTY", Bulletin of the Australian Mathematical Society, 2012 Publication	1%
12	Hee-Sik Kim, Joseph Neggers. "THE SEMIGROUPS OF BINARY SYSTEMS AND SOME PERSPECTIVES", Bulletin of the Korean Mathematical Society, 2008 Publication	1%
13	Bianchi, M.. "Existence of equilibria via Ekeland's principle", Journal of Mathematical Analysis and Applications, 20050515 Publication	1%
14	A. Laurinchikas. "Joint distribution of values of arithmetical functions", Lithuanian	<1%

Mathematical Journal, 1992
Publication

15 Jiming Jiang. "Asymptotic Expansions", Springer Texts in Statistics, 2010 — <1%
Publication

16 Submitted to Cardiff University — <1%
Student Paper

17 Submitted to City University of Hong Kong — <1%
Student Paper

18 Pugachev, . "MOMENTS, CHARACTERISTIC FUNCTIONS, FUNCTIONALS AND CANONICAL EXPANSIONS", Stochastic Systems, 2002. — <1%
Publication

19 Submitted to University of Nottingham — <1%
Student Paper

20 Submitted to Georgia Institute of Technology Main Campus — <1%
Student Paper

21 V. A. Zagrebnov. "Non-Conventional Dynamical Bose Condensation", Physics of Particles and Nuclei, 2021 — <1%
Publication

22 Cao, Y.. "On growth rates of sub-additive functions for semi-flows: Determined and random cases", Journal of Differential Equations, 20061201 — <1%
Publication

23	Janne Kauhanen, Heikki Orelma. "Some theoretical remarks of octonionic analysis", AIP Publishing, 2017 Publication	<1 %
24	Kasparov, G.. "The coarse geometric Novikov conjecture and uniform convexity", Advances in Mathematics, 20061020 Publication	<1 %
25	Submitted to Okinawa Shogaku Senior High School Student Paper	<1 %
26	P. Wynn. "The rational approximation of functions which are formally defined by a power series expansion", Mathematics of Computation, 1960 Publication	<1 %
27	fixedpointtheoryandapplications.springeropen.com Internet Source	<1 %
28	洋輝 菊地, 浩貴 清水, 雄摩 田丸, 憲児 坂本. "MEMS技術を応用した多点法走査型形状測定用センサデバイスの開発（第5報）", 精密工学会学術講演会講演論文集, 2014 Publication	<1 %
29	F. A. Behrend. "Some Remarks on the Construction of Continuous Non-Differentiable Functions", Proceedings of the London Mathematical Society, 01/01/1948 Publication	<1 %

30	M. Denuit, Cl. Lefèvre, Ph. Picard. "Polynomial structures in order statistics distributions", Journal of Statistical Planning and Inference, 2003 Publication	<1 %
31	Tuo-Yeong Lee. "Banach-valued Henstock-Kurzweil integrable functions are McShane integrable on a portion", Mathematica Bohemica, 2005 Publication	<1 %
32	W. H. Young. "On Successions of Integrals and Fourier Series", Proceedings of the London Mathematical Society, 01/01/1913 Publication	<1 %
33	Y. L. Lio, W. J. Padgett. "Asymptotically optimal bandwidth for a smooth nonparametric quantile estimator under censoring", Journal of Nonparametric Statistics, 1992 Publication	<1 %
34	Yu Chuen Wei. "Regularity of “F ” method of summability of sequences", International Journal of Mathematics and Mathematical Sciences, 1989 Publication	<1 %

Exclude quotes	On	Exclude matches	Off
Exclude bibliography	On		

A convergence theorem on the dunford integral

GRADEMARK REPORT

FINAL GRADE

/1000

GENERAL COMMENTS

Instructor

PAGE 1

PAGE 2

PAGE 3

PAGE 4

PAGE 5

PAGE 6

PAGE 7

PAGE 8

PAGE 9