The Maximization of Wealth With Loss Aversion for an Insurance Company

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The insurance company will invest the initial wealth on risk-free assets and risky assets to get the optimal portfolio. Investment decision maker is assumed to be loss aversion. The insurance company want to maximize the expected utility of wealth. In recent years, some researchers interested in optimal investment for a general insurance company. In this paper, the insurance company have a dynamic maximization model, so we translate the dynamic maximization modelinto an equivalent static optimization model with martingale method. Furthemore we solve the static optimization through Lagrange multiplier.
Keywords:insurance company, loss aversion, martingale method.

1. Introduction

Portfolio is investment in some financial instruments that can be traded on the Stock Exchange and Money Market.[1] The expected return refers to the value of a random variable one could expect if the process of finding the random variable could be repeated an infinite number of times. The uncertain condition cause the investor need to establish an optimal portfolio. The aim is to maximizing the expected return and minimize the risk. The concept of portfolio risk was first introduced by Harry M. Markowitz.

The theory of portfolio selection is generally based on the model of expected utility maximization (EUM). In recent years, some theories have been proposed to improve the weaknesses of the theory of EUM, one of them is Tversky at. al with Cumulative Prospect Theory or CPT.[2] The theory of CPT has also been used by Benartzi and Thaler and Bernard & Ghossoub in a static portfolio selection problem.[3,4]Some researchers like Berkelaar et. al studied the problem of dynamic portfolio selection in the CPT.[5] They compared the specifics two parts, namely the utility functions and optimal investment return on the loss averse investors.

Insurance Company is a financial services company with a product that is growing in Indonesia along with the growth of the national economy. Currently the insurance is more often perceived benefits both individual, group, community or business.

Based on previous research, studied the portfolio for the insurer.[6] In this paper, weneed to change the model of maximizing the dynamic becomes static optimization models on the insurance company with the help of martingale method. The surplusprocess of insurance companies is modeled by Lévy process.

2. Problem Formulation

In this study, the problem formulation is how to change the model of maximizing the dynamic becomes static optimization models on the insurance company with the help of martingale method.

3. Methods

In this study, we use the martingale method to translate the model of maximizing the dynamic becomes static optimization models on the insurance company.

Define a martingale as follows.

If $(\Omega, \digamma, P)$ is porobability space. Stochastic process $X: \Omega \times [0, \infty) \to \mathbb{R}$ is denote by X_n called martingale to filtration $\{\digamma_n\}$ if

(1) X_n measurable $-\digamma_n$ for every $n \geq 0$.

(2) $\mathrm{E}(X_n) < \infty$

(3) $E(X_{n+1}|\digamma_n) = X_n$ for every $n \geq 0$

4. Results and Discussion

4.1 Trading Strategies

The insurance company invests in the stock market where assets at $n + 1$ are traded continuouslywhen $[0, T]$. One is a riskless asset with price (S_0) isgiven by

$$dS_0(t) = S_0(t).r(t)dt \tag{4.1}$$

with $S_0(0) = 1$ and $r(t)$ isinterest rate.

While n risk asset with price (S_i) is given by

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dB_j(t)\right] \tag{4.2}$$

$$S_i(0) > 0, j = 1,2, \ldots, n$$

Where $B(t)$ is Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b(t)$ is drift coefficient, and $\sigma(t)$ is volatility.

4.2 Cramér –LundbergProcess

Cramér - Lundberg Process can be used to determine the surplus of insurance company thatgiven by [7]

$$U(t) = x + \int_0^t \alpha(s)ds - S(t) = x + \int_0^t \alpha(s)ds - \sum_{i=1}^{K(t)} Z_i \tag{4.3}$$

where $x > 0$ is initial wealth of insurance company, $\alpha(s) > 0$ is the premium rate at time t, $S(t)$ in a compound Poisson process, and $K(t)$ is number of claims occuring in time interval $[0, t]$.

Stochastic process of (4.3) is $S(t)$ with probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ that can describe by.[7]

(i) $\mathbb{P}$ is a filtration σ and $0 \leq \sigma_1 \leq \sigma_2 \leq \ldots$ is timing of claims payments. If $\sigma_0 = 0$, random variable $T_n = \sigma_n - \sigma_{n-1}$ $(n = 1, 2, \ldots)$ non-negative.

(ii) $K(t) = \sup\{n: \sigma_n \leq t\}$, $t \geq 0$ is number of claims occuring in time interval $[0, t]$. The relationship between time $\{\sigma_0, \sigma_1, \ldots\}$ and counting process $\{K(t), t \geq 0\}$ is given by

$$\{K(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}, \quad n = 0, 1, \ldots$$

(iii) Sequence of non-negative random variable $\{Z_n, n = 1, 2, \ldots\}$ is the amount of claims paid by insurance companies. Sequence of $\{Z_n\}$ is independent, socounting of number of claims up to t is given by

$$S(t) = \sum_{i=1}^{K(t)} Z_i, t \geq 0$$

4.3 Lévy Process

Lévy process has an important role in the financial world. In actuarial, this process is used in the calculation of risk insurance and re-insurance.

Definition4.1

Let$(\Omega, \mathcal{F}, \mathbb{P})$be a filtered probability space,where$\mathcal{F}$is filtration, and stochastic process of$L$is called aLévyprocess if the following conditionsare satisfied.[8]

- Lhas independent increments, i.e$L_t - L_s$is indenpendentof$\mathcal{F}_s$for any $0 \leq s < t < T$.
- Lhas stationary increments,for any$0 \leq s,\, t < T$, the distribution of$L_{t+s} - L$does not depend on$t$.
- $L$is stochastically continuousfor every$0 \leq t \leq T$, and$\varepsilon > 0$ $\lim_{s \to t} P(|L_t - L_s| > \varepsilon) = 0$

Denote$L(t)$is compensatedcompund Poissonprocecs, i.e

$$L(t) \coloneqq S(t) - E(Z) \int_0^t \lambda(s)\, ds$$

Then $L(t)$ is a 1-dimensional compensated pure Lévy process defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $L(t)$ is a martingale.[9]

Let$N$denotePoisson random measure (or jump measure)of$L$and$v$ denotethe Lévy measure that satisfies

$$v(0) = 0, \qquad \int_{\mathbb{R}} (1 + |x|^2\, v(dx) < \infty$$

Intuitively speaking, the Lévy measure describes the expected number of jumps of certain height in a time interval of lenght 1. For the risk process model (3.3), we have $v\,(dx) = \lambda F(dx)$.

L has Lévy decomposition as follows.[10]

$$L(t) = \int_0^t \int_{\mathbb{R}} z\,(N(ds, dz) - v(dz)ds)$$

By using the Lévy process, so that the surplus (3.3) can be given by

$$U(t) = x + \int_0^t c(s)ds - L(t)$$

where$c(s) = \alpha(t) - \lambda(t)E(Z)$.

4.4 Optimal Portfolio Model

In this section we will change the model of maximizing the dynamic becomes static optimization models on the insurance company. The insurer would invest in the $n+1$ assets continuously. We denote $X(t)$ is his wealth at time t. Then $X(t)$ satisfies the following stochastic differential equation:

$$\begin{cases} dX(t) = \big(X(t)r(t) + \pi(t)^T\big(b(t) - r(t)\big) + c(t)\big)dt + \pi(t)^T\sigma(t)dB(t) - dL(t) \\ X(0) = x \end{cases} \tag{4.4}$$

Following utility maximization criterion, the model of maximizing the dynamic of optimal portfolio for an insurer can be formulated as follows:

$$\begin{cases} \max_{\pi \in \Pi} U(X(T)) \\ X(t) \text{satisfies (4.4)} \\ X(t) \geq 0, \forall t \in [0,T] \end{cases} \tag{4.5}$$

According toequation (4.4) we can see that$X(t)$is not amartingale under$\mathbb{P}$, then we need to define the discount factor so that the process $X(t)$ be a martingale. Discount factor is defined as follows.

$$H(t) = \exp\left\{-\int_0^t r(s)\, ds - \frac{1}{2}\int_0^t \|\theta(t)\|^2\, ds - \int_0^t \theta(s)\, dB(s) + \int_0^t \int_{\mathbb{R}} \ln p(s,z) N(ds,dz) + \int_0^t \int_{\mathbb{R}} \big(1 - p(s,z)\big) v(dz) ds\right\} \tag{4.6}$$

where$\int_{\mathbb{R}} \big(1 - p(s,z)\big) v(dz) = c(t)and\theta(t) = \sigma^{-1}[b_i(t) - r(t)]$ismarket price of risk.

We have the following conclusion.

Proposition4.1 If$H(t)$is defined by (4.6) for $t \in [0,T]$, then$H(t)X(t)$is amartingaleunder the probability measure $\mathbb{P}$.

Proof:

According to$X(t)$on (3.4), so we can write $X(t)$in the differential form.

$$\begin{aligned} dX(t) &= \big(X(t)r(t) + \pi(t)^T\big(b(t) - r(t)\big) + c(t)\big)dt + \pi(t)^T\sigma(t)dB(t) - dL(t) \\ &= X(t)r(t)\, dt + \pi(t)^T\big(b(t) - r(t)\big)\, dt + c(t)\, dt + \pi(t)^T\sigma(t)\, dB(t) - dL(t) \\ &= X(t)r(t)\, dt + \pi(t)^T\sigma(t)\left[\left(\frac{b(t) - r(t)}{\sigma(t)}\right) dt + dB(t)\right] + c(t) - dL(t) \\ &= X(t)r(t)\, dt + \pi(t)^T\sigma(t)[\theta\ (t)\, dt + dB(t)] + c(t) - dL(t) \end{aligned}$$

We assumed$dB(t) = \theta\ (t)\, dt + dB(t)$, so

$$dX(t) = X(t)r(t)\, dt + \pi(t)^T\sigma(t)\, dB(t) + c(t) - dL(t) \tag{4.7}$$

Then, from definition of (4.6) we can write $H(t)$in the differential form

$$\begin{aligned} dH(t) &= -r(t)\, H(t)\, dt + \frac{1}{2}\|\theta(t)\|^2\, H(t)\, dt - \frac{1}{2}\|\theta(t)\|^2\, H(t)\, dt \\ &\quad - \theta(t)\, H(t)\, dB(t) + p(t,z)N(dt,dz) - N(dt,dz) - p(t,z)v(dz)dt \\ &\quad + v\ (dz)dt \\ &= -r(t)\, H(t)\, dt - \theta(t)\, H(t)\, dB(t) + (p(t,z) - 1)\ (N(dt,dz) - v(dz)dt)H(t) \\ &= H(t)\big[-r(t)\, dt - \theta(t)\, dB(t) + (p(t,z) - 1)\tilde{N}(dt,dz)\big] \end{aligned} \tag{4.8}$$

To shows thatprocess discounted of $X(t)$is amartingaleunder$\mathbb{P}$, so we can write the differential form as follows

$$\begin{aligned} d\big(H(t)\, X(t)\big) &= dH(t)\, X(t) + H(t)\, dX(t) + d[H(t), X(t)] \\ &= H(t)\, X(t)[-r(t)\, dt - \theta(t)\, dB(t)] \\ &\quad + H(t)[X(t)r(t)\, dt + \pi(t)\sigma(t)\, dB(t)] \\ &\quad + H(t)X(t)(p(t,z) - 1)N(dt,dz) \\ &\quad - H(t)X(t)(p(t,z) - 1)v(dz)dt - H(t)p(t,z)N(dt,dz) \\ &\quad + H(t)p(t,z)v(dz)dt \end{aligned}$$

$$
\begin{aligned}
&= -r(t)\, H(t)\, X(t)\, dt - H(t)\, X(t)\theta(t)\, dB(t) \\
&+ r(t)\, H(t)\, X(t)\, dt + \pi(t)\, \sigma(t)\, H(t)dB(t) \\
&+ H(t)N(dt, dz)[X(t)(p(t,z) - 1) - p(t,z)] \\
&- H(t)v(dz)dt[X(t)(p(t,z) - 1) - p(t,z)] \\
&= -H(t)\, X(t)\theta(t)\, dB(t) + \pi(t)\, \sigma(t)\, H(t)dB(t) \\
&+ H(t)[X(t)(p(t,z) - 1) - p(t,z)](N(dt,dz) - v(dz)dt) \\
&= H(t)[-X(t)\theta(t) + \pi(t)\, \sigma(t)]\, dB(t) + H(t)[X(t)(p(t,z) - 1) - p(t,z)]\tilde{N}(dt,dz)
\end{aligned}
\tag{4.9}
$$

We pray for integration on both sides of an equation (4.9) and get

$$
\begin{aligned}
H(t)X(t) = X(0) &+ \int_0^t H(s)[-X(s)\theta(s) + \pi(s)\sigma(s)]\ dB(s) \\
&+ \int_0^t H(t)[X(t)(p(t,z) - 1) - p(t,z)]\tilde{N}(dt,dz)
\end{aligned}
\tag{4.10}
$$

Based on the above equation, since every term is an integral Ito, then $H(t)X(t)$ is a martingale under the probability measure $\mathbb{P}$.

■

Then, we need to change the model of maximizing the dynamic (4.5) becomes static optimization modelsthat satisfied Theorem 4.1.

Theorem4.1 Let$\eta \geq 0$be an$\mathcal{F}_t$-measurable random variable, then for a given initial wealth$x$satisfying$\mathbb{E}[H(T)\eta] = x$there exist a portfolio process such that$\pi(t) \in \Pi$, $t \in [0,T]$and$X^{\pi}(T) = \eta$.

Proof:

Define a negative martingale$M(t) = \mathbb{E}[H(T)\eta|\mathcal{F}_t]$.

$M(t)$is$\widetilde{\mathbb{P}}$-*martingale* so according to the martingale representation theorem, there exist a predictable process $\psi(t)\colon \Omega \times [0,T] \longmapsto \mathbb{R}and\varphi(t)\colon \Omega \times [0,T] \longmapsto \mathbb{R}^n,\ 0 \leq t \leq T$.[11]The theorem is satisfying

$$
\int_0^T \|\varphi(s)\|^2 ds < \infty\ , a.s
$$

$$
\int_0^T \int_{\mathbb{R}} |\psi(t,z)|^2\ v(dz)dt < \infty\,, a.s
$$

Such that

$$
\begin{aligned}
M(t) &= M_0 + \int_0^t \varphi(s)\ dB(s) + \int_0^T \int_{\mathbb{R}} \psi(t,z)\tilde{N}(ds,dz) \\
&= x + \int_0^t \varphi(s)\ dB(s) + \int_0^T \int_{\mathbb{R}} \psi(t,z)\tilde{N}(ds,dz)
\end{aligned}
\tag{4.11}
$$

Comparing $dB(t)$-term, and $\tilde{N}(dt,dz)$-term respectively in (4.11) and (4.10) on the rigth side, it is reasonable conjecture

$$
X(0) = M(0) = \mathbb{E}[H(T)\eta] \tag{4.12}
$$

$$
\pi(t) = (\sigma(t)^T)^{-1}[\varphi(t)H(t)^{-1} + \theta(t)X(t)] \tag{4.13}
$$

$$
\psi(t,z) = H(t)[X(t)(p(t,z) - 1) - p(t,z)] \tag{4.14}
$$

while on the left side, we have

$$
H(t)X(t) = M(t) = \mathbb{E}[H(T)\eta|\mathcal{F}_t]
$$

$$
H(t)X(t) = H(T)\eta
$$

$$X(t) = \eta \tag{4.15}$$

Thenwe need to check if the process $\pi(t)$ defined in (4.13) is an admissible portfolio. Let $(\sigma(t)^T)^{-1} = (c_{ij}(t))_{n\times n}$,$\varphi(t) = \left(\varphi_1(t), \dots, \varphi_n(t)\right)^T$, and $\theta(t) = \left(\theta_1(t), \dots, \theta_n(t)\right)^T$.
We define some notations:

$$\|f(t)\|_\infty = \max_{0\leq t\leq T}|f(t)| \text{ dan } \|f(t)\|_2 = \left(\int_0^T |f(t)|^2 dt\right)^{\frac{1}{2}}$$

We have,

$$\begin{aligned}
\int_0^T |\pi_i(t)|dt &= \int_0^T |(\sigma(t)^T)^{-1}[\varphi(t)H(t)^{-1} + \theta(t)X(t)]|dt \\
&= \int_0^T \left|(c_{ij}(t))_{n\times n}\left[\varphi_j(t)H(t)^{-1} + \theta_j(t)X(t)\right]\right|dt \\
&\leq \int_0^T \left|(c_{ij}(t))_{n\times n}\varphi_j(t)H(t)^{-1}\right|dt + \int_0^T \left|(c_{ij}(t))_{n\times n}\theta_j(t)X(t)\right| dt \\
&\leq \int_0^T \left|H(t)^{-1}\sum_{j=1}^{n} c_{ij}(t)\,\varphi_j(t)\right| dt + \int_0^T \left|X(t)\sum_{j=1}^{n} c_{ij}(t)\,\theta_j(t)\right| dt \\
&\leq \|H^{-1}\|_\infty \left\|c_{ij}(t)\right\|_\infty \left\|\varphi_j(t)\right\|_2 + \|X(t)\|_\infty \left\|c_{ij}(t)\right\|_\infty \left\|\theta_j(t)\right\|_2
\end{aligned}$$

$< \infty$ a.s

■

The last inequality follows from the uniformly bounded conditions.

According to Theorem 4.1, any $\mathcal{F}_t$-measurable random variable $\eta \geq 0$ with $\mathbb{E}[H(T)\eta] = x$ can be financed via trading an admissible portfolio π such that $X^\pi(T) = \eta$. So to determine the optimal portfolio process $\pi^*(t)$ in the dynamic maximization models (4.5), which depends on the time variable t, it is sufficient to maximize over all possible random variable η's. That is to say, the dynamic maximization models (4.5) is equivalent to the following static optimization models:

$$\begin{cases} \max\limits_{\eta\geq 0} \mathbb{E}[U(\eta)] \\ s.t\ \ \mathbb{E}[H(T)\eta] \leq x \end{cases} \tag{4.15}$$

4. Conclusion

In conclusion, this study shows thatwith martingale method we have optimal portfolio process $\pi^*(t)$ in the dynamic maximization models, which depends on the time variable t, it is sufficient to maximize over all possible random variable η's. It means, the dynamic maximization problem is equivalent to the following static optimization models. Such that, the static optimization models on the insurance company is

$$\begin{cases} \max\limits_{\eta\geq 0} \mathbb{E}[U(\eta)] \\ s.t\ \ \mathbb{E}[H(T)\eta] \leq x \end{cases}$$

References

1. S. Mohamad. *Pasar Modal dan Manajemen Portofolio.*285**(2006)**

2. T. Amos and K. Daniel.*J. Risk Uncertain.* 5. 297–323**(1992)**

3. B. Shlomo and T. H. Rhicard. *The Quarterly Journal of Economics.* 110. 73-92**(1995)**

4. B. Carole and G. Mario. *Math. Financ. Econ.* 2. 277-306 **(2010)**

5. B. Arjan andK. Roy. *Rev. Econ. Stat.*86. 973–987**(2004)**

6. G. Wenjing. *Insurance: Mathematics and Economics*. 58. 217–222**(2014)**

7. M. D. Leda. *Insurance Risk Theory*.**(2010)**

8. P. Antonis. *Mathematics Subject Classification.* 60G51,60E07,60G44,91B28 **(2000)**

9. S. E. Shreve. *Springer-Verlag, Berlin.***(2004)**

10. O. Bernt and S. Agnes. *In: Universitext, Springer-Verlag, Heidelberg, New York* **(2005)**

11. C. Rama and T. Peter. *Chapman and Hall/CRC, BR* **(2004)**