Application of Robust Linear Quadratic Control for Inventory System with Unknown Demand: Single Product Case

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***Abstract*— In this paper, a dynamical model of single product inventory system with unknown demand in a linear state space equation with unknown parameter for inventory control purposes was formulated. An existing control method, robust linear quadratic regulator (RLQR), was applied to control the inventory level by generating the optimal purchasing product volume so that the product stock follows a reference trajectory with minimal cost. The result of the performed numerical experiments showed that the optimal purchasing product volume was determined for every time period and the product stock was closed to the given trajectory level desired by the decision maker.**

Keywords— inventory control; inventory system; robust LQR supply chain management; unknown demand; unknown parameter

I. Introduction

Supply chain managerial strategies have been developed and applied widely in many areas. In supply chain, the strategy focuses in the relationships for all parties in term to optimize the process and the gained outcome [1]. All parties which are supplier, manufacturer, carrier, warehouse, retailer and at the end of chain, customer are containing management process that can be optimized efficiently [2]. Commonly the manufacturer, warehouse and retailer are have an inventory system to store their raw material/product that will be used to satisfy the demand in the next time periods. Since storing the product in an inventory system will be causing some holding cost then optimization/controlling will be needed to find the optimal decision on it.

Some decision makers like in retail parties, commonly wish to control the product stock level in his inventory such that it will follow a reference point/trajectory that will be decided by the manager or decision maker. In order to control the inventory level efficiently, some researchers have been developed the mathematical model of the dynamics of the inventory level and used some theoretical approaches like control theory and optimization to find the optimal decision. The pioneer and basic model in a dynamical equation of the inventory system was formulated as a linear discrete time invariant state space where the linear quadratic controller was applied to find the optimal decision [3]. Another approach was found in an optimization model to find the optimal policy of inventory system in single period and two suppliers [4].

Some researchers were focused not only on inventory controlling but also integrating with other components in the chain like supplier selection [5]–[7]. There are many method of solution approaches found in the literatures like particle swarm optimization and predictive control based on quadratic programming [8], [9].

In control system theory, there are many control methods that can be used to obtain the optimal control or optimal decision for many problems in the industrial applications. Some modern control methods that have been developed to be applied in industrial automation are robust control methods like robust linear quadratic control method. The term “robust” means that these kind of control methods are able to handle some uncertain or unknown parameters or disturbances. Robust LQR was applied in so many problems like, for some recent applications, inverted pendulum controlling [10]laser beam shaping [11], controlling of vehicle [12] and stall flutter suppression control problem [13]. For further and more complex applications, robust LQR has been developed by integrating it with other control methods or approaches like Neural Networks for formation flying of satellite controlling [14], to be used for bilinear systems and multi-agent controlling [15], [16], adaptive controlling for helicopter [17], predictive controller [18] and genetic algorithm based controller for fuel cell voltage control problem [19].

Most of the mentioned papers above were solved the inventory control problem with well-known demand value. In this article, a dynamical equation approach to control a single product case of inventory or warehouse system in a linear dynamical state space equation with unknown demand will be formulated for inventory level control purposes. To find the optimal outcome that will be obtained by the optimal decision which is how much the purchased product volume for each review time cycle, an existing control method which is robust linear quadratic control will be applied. To illustrate and evaluate the solution of the problem, a numerical simulation will be performed.

II. Materials and Methods

A. Problem Definition

Suppose a single product case of inventory or warehouse system considering single supplier where the dynamics of the stored product level is depending on the arriving ordered/purchased product and its demand value. The demand value is unknown for all optimization/controlling periods of time. This unknown demand value is reputed as a random variable with lower and upper bounded. The decision maker (DM) wants to control the product volume in

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the inventory so that it follows a set point/trajectory decided by the DM with minimal effort which could be cost. Then, the problem is how to find the optimal decision i.e. how much the product unit that must be ordered at any time period such that the product stock unit in the inventory will follow the desired set point as close as possible and the demand is expected to be satisfied.

B. Assumptions

The formulated model is satisfied the following assumptions:

1. The product is the same in size;
2. There is no damaged product in the delivery process;
3. The product will be not expired over all of controlling time periods;
4. There is no shortage product which means that all of the ordered product will be delivered;
5. The inventory level is initially empty.

C. Mathematical Model

Suppose an inventory control problem defined in Sub-section II.A where assumptions in Sub-section II.B are hold. Let $(T\cdot k), k=0,1,2,...$ is denoting the review time period, T is denoting time instant (e.g. hour, week, month, etc), $y(T\cdot k)$ is denoting the level of the stored product in the inventory at review time period $(T\cdot k)$ and $d(Tk)$ is denoting the demand value of the product at time period Tk.. By assumption (5), for $Tk \le 0$ the product stock will be $y(Tk)=0$. Furthermore, if $u(Tk)$ is denoting the arriving purchased/ordered product at k and n_p is denoting the time delay between ordering and arriving product, then the dynamics of the product stock for any $k \ge 0$ may be modelled as

$$\begin{aligned} y(Tk) &= u(0)+u(T\cdot 1)+u(T\cdot 2)+...+u\left(T\cdot\left(k-n_p-1\right)\right) \\ &\quad -d(0)-d(T\cdot 1)-...-d\left(T\cdot(k-1)\right) \\ &= \sum_{j=0}^{T\cdot\left(k-n_p-1\right)} u(T\cdot j)-\sum_{j=0}^{T\cdot(k-1)} d(T\cdot j). \end{aligned} \tag{1}$$

Let for any time period $T\cdot k>0$,

$y(T\cdot k)=x_1(T\cdot k)$,

$x_i(T\cdot k)=u\left(T\cdot(k-n_p+i)\right)$ for $i=2,3,...,n_p+1$,

$n=n_p+1$.

then the dynamical equation **Error! Reference source not found.** can be rewritten as

$$\left.\begin{aligned} x_0\left(T(k+1)\right) &= x_0(Tk), x_0(0)=1, \\ x_1\left(T(k+1)\right) &= x_1(Tk)+x_2(Tk) \\ &\quad -d(Tk)x_0(Tk) \\ x_2\left(T(k+1)\right) &= x_3(Tk) \\ x_3\left(T(k+1)\right) &= x_4(Tk) \\ &\vdots \\ x_n\left(T(k+1)\right) &= u(Tk). \end{aligned}\right\} \tag{2}$$

Furthermore, let

$x(Tk)=\left[x_0(Tk), x_1(Tk), x_2(Tk), ..., x_n(Tk)\right]'$,

then equation **Error! Reference source not found.** may be rewritten as a linear discrete time state space as follows

$$\begin{bmatrix} x_0\left(T(k+1)\right) \\ x_1\left(T(k+1)\right) \\ x_2\left(T(k+1)\right) \\ x_3\left(T(k+1)\right) \\ \vdots \\ x_n\left(T\cdot(k+1)\right) \end{bmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}(Tk) + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -d(Tk) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}\right] \cdot \begin{bmatrix} x_0(Tk) \\ x_1(Tk) \\ x_2(Tk) \\ x_3(Tk) \\ \vdots \\ x_n(Tk) \end{bmatrix} \tag{3}$$

or it can be rewritten as

$$\begin{aligned} x\left(T\cdot(k+1)\right) &= \left(F(T\cdot k)+\delta F(T\cdot k)\right)x(T\cdot k) \\ &\quad +\left(G(T\cdot k)+\delta G(T\cdot k)\right)u(T\cdot k), \\ k &= 0,1,2,\ldots \end{aligned} \tag{4}$$

where

$$F(T\cdot k)=\begin{pmatrix}1&0&0&0&\cdots\\0&1&1&0&\cdots\\0&0&0&1&\cdots\\0&0&0&0&\ddots&\vdots\\\vdots&\vdots&\vdots&\vdots&\ddots\\0&0&0&0&\cdots\end{pmatrix}\quad (T\cdot k)=\begin{bmatrix}0\\0\\0\\0\\\vdots\\1\end{bmatrix},$$

$$\delta F(T\cdot k)=\begin{pmatrix}0&0&0&0&\cdots\\-d(T\cdot k)&0&0&0&\cdots\\0&0&0&0&\cdots\\0&0&0&0&\ddots&\vdots\\\vdots&\vdots&\vdots&\vdots&\ddots\\0&0&0&0&\cdots\end{pmatrix}$$

$$,\delta G(T\cdot k)=\begin{bmatrix}0\\0\\0\\0\\\vdots\\0\end{bmatrix}$$

where vector $x(T\cdot k)\in\mathbb{R}$ denotes the state value, vector $u(T\cdot k)\in\mathbb{R}$ denotes the input value, matrices $F(T\cdot k)\in\mathbb{R}$ and $G(T\cdot k)\in\mathbb{R}$ are the nominal parameter, and

$$\begin{aligned}&\left[\delta F(T\cdot k),\delta G(T\cdot k)\right]=\\&H(T\cdot k)\Delta(T\cdot k)\left[E_F(T\cdot k),E_G(T\cdot k)\right],k=0,1,2,\ldots\end{aligned}\tag{5}$$

are uncertainty matrices with dimension

$H(T\cdot k)\in\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$.

D. Robust LQR (Review)

Consider a discrete time linear system as follows

$$\begin{aligned}x\left(T\cdot(k+1)\right)&=\left(F(T\cdot k)+\delta F(T\cdot k)\right)x(T\cdot k)\\&+\left(G(T\cdot k)+\delta G(T\cdot k)\right)u(T\cdot k),\end{aligned}\tag{6}$$

for $k=0,1,\ldots,N$ where

$x(T\cdot k)\in\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$

and $G(T\cdot k)\in\mathbb{R}$ are state vector, input vector, matrix coefficient (known for each *k*) for *x* and matrix coefficient (known for each *k*) for *u* respectively. Furthermore, matrices $\delta F(T\cdot k)$ and $\delta G(T\cdot k)$ are uncertain matrices that for any *k,* they can be approached by equation

$$\begin{cases}\delta F(T\cdot k)=H(T\cdot k)\Delta(T\cdot k)E_F(T\cdot k)\\\delta G(T\cdot k)=H(T\cdot k)\Delta(T\cdot k)E_G(T\cdot k)\end{cases}$$

where $H(T\cdot k)\in\mathbb{R}$ $\Delta(T\cdot k)\in\mathbb{R}$ with $\|\Delta(T\cdot k)\|\le 1$, $E_F(T\cdot k)\in\mathbb{R}$ and $E_G\in\mathbb{R}$ are certain (known) matrices for any *k*. For simplicity purposes, let $T\cdot k=k$. The RLQR method optimizes the objective function

$$\min_{x(k+1),u(k)}\ \max_{\delta G(k),\delta F(k)}\ \tilde{}\tag{7}$$

where

$$\begin{aligned}\tilde{}\quad &J\left(x(k+1),u(k),\delta F(k),\delta G(k),\mu\right)\\&=\begin{bmatrix}x(k+1)\\u(k)\end{bmatrix}^T\begin{bmatrix}P(k+1)&0\\0&R(k)\end{bmatrix}\begin{bmatrix}x(k+1)\\u(k)\end{bmatrix}\\&+\Theta^T\begin{bmatrix}Q(k)&0\\0&\mu I\end{bmatrix}\Theta,\\\Theta&=\left(\begin{bmatrix}0&0\\I&-G(k)\end{bmatrix}+\begin{bmatrix}0&0\\I&-\delta G(k)\end{bmatrix}\right)\begin{bmatrix}x(k+1)\\u(k)\end{bmatrix}\\&-\begin{bmatrix}-I\\F(k)\end{bmatrix}x(k)-\begin{bmatrix}0\\\delta F(k)\end{bmatrix}x(k),\end{aligned}$$

$P(k),Q(k)\succ$ $\succ$ for any k are real valued matrices which called as weighting matrices decided by the decision maker and μ denotes a real number as penalty parameter value. The optimal input $u^*(k)$ and the corresponding state $x^*(k+1)$ is derived from

$$\begin{bmatrix}x^*(k+1)\\u^*(k)\end{bmatrix}=\begin{bmatrix}L(k)\\K(k)\end{bmatrix}x^*(k)\tag{8}$$

where $L(k)$ and $K(k)$ is derived from backward calculation

$$\begin{bmatrix}L(k)\\K(k)\\P(k)\end{bmatrix}=\Sigma^T\Omega\Psi,k=N,N-1,\ldots,0,\tag{9}$$

where

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & 0 & F(k) \\ 0 & 0 & E_F(k) \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 0 \\ 0 \\ -I \\ F(k) \\ E_F(k) \\ 0 \\ 0 \end{bmatrix},$$

$$\Omega_1 = \begin{bmatrix} P^{-1}(k+1) & 0 & 0 \\ 0 & R^{-1}(k) & 0 \\ 0 & 0 & Q^{-1}(k) \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Omega_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_4 = \begin{bmatrix} 0 & 0 & I & -G(k) \\ 0 & 0 & 0 & -E_G(k) \\ I & 0 & 0 & 0 \\ -G^T(k) & -E_G^T(k) & 0 & 0 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix},$$

with some given initial state $x(0)$, $P(N+1) \succ$.

III. Computational Simulation

Problem **Error! Reference source not found.** was simulated with $n_p = 2$ and n=3 where the dynamical system of this problem can be formulated as

$$x(k+1) = \begin{bmatrix} x_0(k+1) \\ x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{bmatrix} x_0(k) \\ x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k), \quad (10)$$

where $x(0) = [1,100,0,0]^T$ and

$$F(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \delta F(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G(k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \delta G(k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

TABLE I. Data Parameters

Parameter	Q	R	u_{min}	u_{max}	y_{min}	y_{max}
Value	I	I	0	400	0	500

The data parameters are given in TABLE I. We have simulated this system for 80 review time periods. The set point of the inventory level is 20. The demand value is randomly generated with normal distribution with mean 20 and variance 5. The optimal input values i.e. the optimal purchased product volume for all review time periods are shown in Fig. 1.

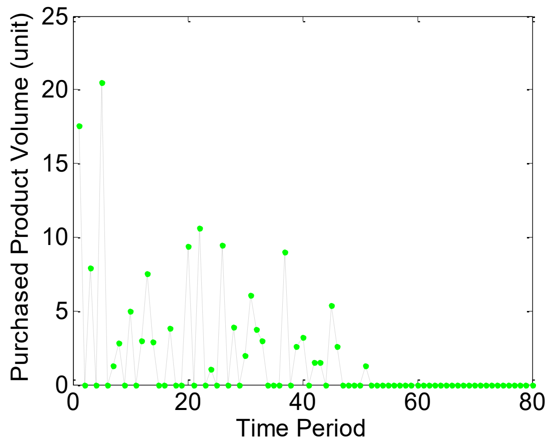

Fig. 1. The optimal purchasing product volume

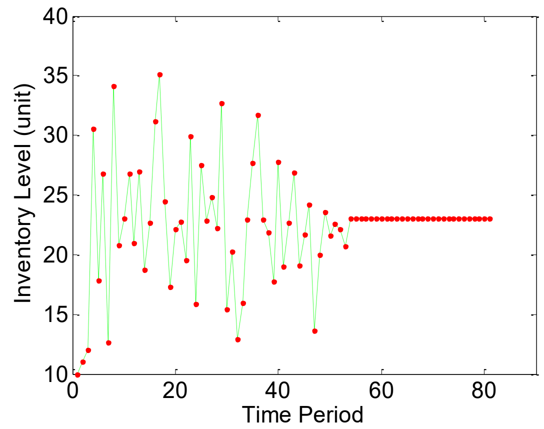

Fig. 2. Inventory level and its set point level

Fig 2 describes the inventory level or product stock stored in the warehouse for review time period 0 to 80. For time periods 0 to about 50, the demand value was generated randomly where the optimal purchased product volume is shown in Fig. 1. It can be observed from Fig. 2 that the inventory level was fluctuated at a range in about 20 units that was accommodated the demand value. For review time periods about 50 to 80, the demand value was simulated to be 0 then the purchased product volume was determined as 0 (there is no demand to be satisfied).

IV. Conclusions

In this article, the robust linear quadratic regulator was applied for inventory controlling of.single product case of warehouse system considering unknown demand. The controller was successfully generated the optimal decision which is the optimal purchasing product volume for all simulation time periods and the product stock level was nearby and sufficiently closed to the given set point as desired by the decision maker.

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